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NOTICE OF ALLOWANCE AND FEE(S) DUE

02292 7590 04/11/2005

BIRCH STEWART KOLASCH & BIRCH
PO BOX 747
FALLS CHURCH, VA 22040-0747

EXAMINER	
NYALLEY, LANSANA	
ART UNIT	PAPER NUMBER
1621	

DATE MAILED: 04/11/2005

APPLICATION NO.	FILING DATE	FIRST NAMED INVENTOR	ATTORNEY DOCKET NO.	CONFIRMATION NO.
10/829,193	04/22/2004	Ayumu Kiyomori	0171-1086PUS1	5551

TITLE OF INVENTION: PREPARATION OF SILYL KETENE ACETAL AND DISILYL KETENE ACETAL

APPLN. TYPE	SMALL ENTITY	ISSUE FEE	PUBLICATION FEE	TOTAL FEE(S) DUE	DATE DUE
nonprovisional	NO	$1400	$300	$1700	07/11/2005

THE APPLICATION IDENTIFIED ABOVE HAS BEEN EXAMINED AND IS ALLOWED FOR ISSUANCE AS A PATEN PROSECUTION ON THE MERITS IS CLOSED. THIS NOTICE OF ALLOWANCE IS NOT A GRANT OF PATENT RIGHT THIS APPLICATION IS SUBJECT TO WITHDRAWAL FROM ISSUE AT THE INITIATIVE OF THE OFFICE OR UPO PETITION BY THE APPLICANT. SEE 37 CFR 1.313 AND MPEP 1308.

THE ISSUE FEE AND PUBLICATION FEE (IF REQUIRED) MUST BE PAID WITHIN THREE MONTHS FROM TH MAILING DATE OF THIS NOTICE OR THIS APPLICATION SHALL BE REGARDED AS ABANDONED. TH STATUTORY PERIOD CANNOT BE EXTENDED. SEE 35 U.S.C. 151. THE ISSUE FEE DUE INDICATED ABOV REFLECTS A CREDIT FOR ANY PREVIOUSLY PAID ISSUE FEE APPLIED IN THIS APPLICATION. THE PTOL-85B (O AN EQUIVALENT) MUST BE RETURNED WITHIN THIS PERIOD EVEN IF NO FEE IS DUE OR THE APPLICATION WIL BE REGARDED AS ABANDONED.

HOW TO REPLY TO THIS NOTICE:

I. Review the SMALL ENTITY status shown above.

If the SMALL ENTITY is shown as YES, verify your current SMALL ENTITY status:

A. If the status is the same, pay the TOTAL FEE(S) DUE shown above.

B. If the status above is to be removed, check box 5b on Part B - Fee(s) Transmittal and pay the PUBLICATION FEE (if required) and twice the amount of the ISSUE FEE shown above, or

If the SMALL ENTITY is shown as NO:

A. Pay TOTAL FEE(S) DUE shown above, or

B. If applicant claimed SMALL ENTITY status before, or is n claiming SMALL ENTITY status, check box 5a on Part B - Fee Transmittal and pay the PUBLICATION FEE (if required) and the ISSUE FEE shown above.

II. PART B - FEE(S) TRANSMITTAL should be completed and returned to the United States Patent and Trademark Office (USPTO) w your ISSUE FEE and PUBLICATION FEE (if required). Even if the fee(s) have already been paid, Part B - Fee(s) Transmittal should completed and returned. If you are charging the fee(s) to your deposit account, section "4b" of Part B - Fee(s) Transmittal should completed and an extra copy of the form should be submitted.

III. All communications regarding this application must give the application number. Please direct all communications prior to issuance Mail Stop ISSUE FEE unless advised to the contrary.

IMPORTANT REMINDER: Utility patents issuing on applications filed on or after Dec. 12, 1980 may require payment maintenance fees. It is patentee's responsibility to ensure timely payment of maintenance fees when due.

PART B - FEE(S) TRANSMITTAL

Complete and send this form, together with applicable fee(s), to: **Mail** **Mail Stop ISSUE FEE**
Commissioner for Patents
P.O. Box 1450
Alexandria, Virginia 22313-1450
or Fax (703) 746-4000

INSTRUCTIONS: This form should be used for transmitting the ISSUE FEE and PUBLICATION FEE (if required). Blocks 1 through 5 should be completed wh appropriate. All further correspondence including the Patent, advance orders and notification of maintenance fees will be mailed to the current correspondence address indicated unless corrected below or directed otherwise in Block 1, by (a) specifying a new correspondence address; and/or (b) indicating a separate "FEE ADDRESS" maintenance fee notifications.

CURRENT CORRESPONDENCE ADDRESS (Note: Use Block 1 for any change of address)

02292 7590 04/11/2005

BIRCH STEWART KOLASCH & BIRCH
PO BOX 747
FALLS CHURCH, VA 22040-0747

Note: A certificate of mailing can only be used for domestic mailings of Fee(s) Transmittal. This certificate cannot be used for any other accompany papers. Each additional paper, such as an assignment or formal drawing, m have its own certificate of mailing or transmission.

Certificate of Mailing or Transmission

I hereby certify that this Fee(s) Transmittal is being deposited with the Un States Postal Service with sufficient postage for first class mail in an envel addressed to the Mail Stop ISSUE FEE address above, or being facsim transmitted to the USPTO (703) 746-4000, on the date indicated below.

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10/829,193	04/22/2004	Ayumu Kiyomori	0171-1086PUS1	5551

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APPLN. TYPE	SMALL ENTITY	ISSUE FEE	PUBLICATION FEE	TOTAL FEE(S) DUE	DATE DUE
nonprovisional	NO	$1400	$300	$1700	07/11/2005

EXAMINER	ART UNIT	CLASS-SUBCLASS
NYALLEY, LANSANA	1621	556-446000

1. Change of correspondence address or indication of "Fee Address" (37 CFR 1.363).
   - ☐ Change of correspondence address (or Change of Correspondence Address form PTO/SB/122) attached.
   - ☐ "Fee Address" indication (or "Fee Address" Indication form PTO/SB/47; Rev 03-02 or more recent) attached. **Use of a Customer Number is required.**

2. For printing on the patent front page, list
   (1) the names of up to 3 registered patent attorneys or agents OR, alternatively,
   (2) the name of a single firm (having as a member a registered attorney or agent) and the names of up to 2 registered patent attorneys or agents. If no name is listed, no name will be printed.
   1 __________
   2 __________
   3 __________

3. ASSIGNEE NAME AND RESIDENCE DATA TO BE PRINTED ON THE PATENT (print or type)

PLEASE NOTE: Unless an assignee is identified below, no assignee data will appear on the patent. If an assignee is identified below, the document has been filed recordation as set forth in 37 CFR 3.11. Completion of this form is NOT a substitute for filing an assignment.

(A) NAME OF ASSIGNEE

(B) RESIDENCE: (CITY and STATE OR COUNTRY)

Please check the appropriate assignee category or categories (will not be printed on the patent) : ☐ Individual ☐ Corporation or other private group entity ☐ Governm

4a. The following fee(s) are enclosed:
- ☐ Issue Fee
- ☐ Publication Fee (No small entity discount permitted)
- ☐ Advance Order - # of Copies __________

4b. Payment of Fee(s):
- ☐ A check in the amount of the fee(s) is enclosed.
- ☐ Payment by credit card. Form PTO-2038 is attached.
- ☐ The Director is hereby authorized by charge the required fee(s), or credit any overpayment Deposit Account Number __________ (enclose an extra copy of this form).

5. **Change in Entity Status** (from status indicated above)
   - ☐ a. Applicant claims SMALL ENTITY status. See 37 CFR 1.27.
   - ☐ b. Applicant is no longer claiming SMALL ENTITY status. See 37 CFR 1.27(g)(2).

The Director of the USPTO is requested to apply the Issue Fee and Publication Fee (if any) or to re-apply any previously paid issue fee to the application identified above.
NOTE: The Issue Fee and Publication Fee (if required) will not be accepted from anyone other than the applicant; a registered attorney or agent; or the assignee or other part interest as shown by the records of the United States Patent and Trademark Office.

Authorized Signature __________ Date __________

Typed or printed name __________ Registration No. __________

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NYALLEY, LANSANA	
ART UNIT	**PAPER NUMBER**
1621	

DATE MAILED: 04/11/2005

Determination of Patent Term Adjustment under 35 U.S.C. 154 (b)
(application filed on or after May 29, 2000)

The Patent Term Adjustment to date is 0 day(s). If the issue fee is paid on the date that is three months after t mailing date of this notice and the patent issues on the Tuesday before the date that is 28 weeks (six and a ha months) after the mailing date of this notice, the Patent Term Adjustment will be 0 day(s).

If a Continued Prosecution Application (CPA) was filed in the above-identified application, the filing date th determines Patent Term Adjustment is the filing date of the most recent CPA.

Applicant will be able to obtain more detailed information by accessing the Patent Application Information Retriev (PAIR) WEB site (http://pair.uspto.gov).

Any questions regarding the Patent Term Extension or Adjustment determination should be directed to the Office Patent Legal Administration at (571) 272-7702. Questions relating to issue and publication fee payments should directed to the Customer Service Center of the Office of Patent Publication at (703) 305-8283.

Notice of Allowability	Application No. 10/829,193	Applicant(s) KIYOMORI ET AL.	
	Examiner Lansana Nyalley	Art Unit 1621	

-- The MAILING DATE of this communication appears on the cover sheet with the correspondence address--

All claims being allowable, PROSECUTION ON THE MERITS IS (OR REMAINS) CLOSED in this application. If not included herewith (or previously mailed), a Notice of Allowance (PTOL-85) or other appropriate communication will be mailed in due course. **THIS NOTICE OF ALLOWABILITY IS NOT A GRANT OF PATENT RIGHTS.** This application is subject to withdrawal from issue at the initiative of the Office or upon petition by the applicant. See 37 CFR 1.313 and MPEP 1308.

1. ☒ This communication is responsive to *THE ARGUMENT FILED ON 01-18-2005*.
2. ☒ The allowed claim(s) is/are *1-5 and 7-15*.
3. ☐ The drawings filed on _____ are accepted by the Examiner.
4. ☒ Acknowledgment is made of a claim for foreign priority under 35 U.S.C. § 119(a)-(d) or (f).
   a) ☒ All b) ☐ Some* c) ☐ None of the:
   1. ☒ Certified copies of the priority documents have been received.
   2. ☐ Certified copies of the priority documents have been received in Application No. _____.
   3. ☐ Copies of the certified copies of the priority documents have been received in this national stage application from the International Bureau (PCT Rule 17.2(a)).

   * Certified copies not received: _____.

Applicant has THREE MONTHS FROM THE "MAILING DATE" of this communication to file a reply complying with the requirements noted below. Failure to timely comply will result in ABANDONMENT of this application.
THIS THREE-MONTH PERIOD IS NOT EXTENDABLE.

5. ☐ A SUBSTITUTE OATH OR DECLARATION must be submitted. Note the attached EXAMINER'S AMENDMENT or NOTICE OF INFORMAL PATENT APPLICATION (PTO-152) which gives reason(s) why the oath or declaration is deficient.
6. ☐ CORRECTED DRAWINGS (as "replacement sheets") must be submitted.
   (a) ☐ including changes required by the Notice of Draftsperson's Patent Drawing Review (PTO-948) attached
   1) ☐ hereto or 2) ☐ to Paper No./Mail Date _____.
   (b) ☐ including changes required by the attached Examiner's Amendment / Comment or in the Office action of Paper No./Mail Date _____.

   Identifying indicia such as the application number (see 37 CFR 1.84(c)) should be written on the drawings in the front (not the back) of each sheet. Replacement sheet(s) should be labeled as such in the header according to 37 CFR 1.121(d).
7. ☐ DEPOSIT OF and/or INFORMATION about the deposit of BIOLOGICAL MATERIAL must be submitted. Note the attached Examiner's comment regarding REQUIREMENT FOR THE DEPOSIT OF BIOLOGICAL MATERIAL.

Attachment(s)

1. ☐ Notice of References Cited (PTO-892)
2. ☐ Notice of Draftperson's Patent Drawing Review (PTO-948)
3. ☐ Information Disclosure Statements (PTO-1449 or PTO/SB/08), Paper No./Mail Date _____
4. ☐ Examiner's Comment Regarding Requirement for Deposit of Biological Material
5. ☐ Notice of Informal Patent Application (PTO-152)
6. ☐ Interview Summary (PTO-413), Paper No./Mail Date _____.
7. ☒ Examiner's Amendment/Comment
8. ☒ Examiner's Statement of Reasons for Allowance
9. ☐ Other _____.

Examiners Amendment.

An examiner's amendment to the record appears below. Should the changes and/or additions be unacceptable to applicant, an amendment may be filed as provided by 37 CFR 1.312. To ensure consideration of such an amendment, it MUST be submitted no later than the payment of the issue fee.

Authorization for this examiner's amendment was given in a telephone interview with Mr. Gerald Murphy on 04-05-05.

--Claim 6 is cancelled.

--In claim 7, line 3, before the word "and", delete the letter "a."

Reasons For Allowance.

The following is an Examiner's reason for allowance.

Applicants' argument filed on 01-18-05 to over come the 35 USC 103 rejection made in the office action of 09-28-04 is found convincing.

Additionally, Applicants' process for preparing a silyl ketene acetal according to claim one of the instant application is neither obvious nor suggested over the prior art of record.

Dinh et. al teach a process of preparing a silyl ketene acetal by reacting an α,β-unsaturated hydrosilane or hydrosiloxane in the presence of Rhcl(di-tert-butylsulfide)2 catalyst. Dinh et. al do not teach the use of a tris(pentafluorophenyl)borane.

The selection of a tris(pentafluorophenyl)borane catalyst in this reaction would not have been suggested to one of ordinary skill in the art at the time the invention was made.

Thus, the Examiner finds claims 1-5, 7-15 allowable.

Any inquiry concerning this communication or earlier communications from the examiner should be directed to Lansana Nyalley whose telephone number is 571,272,0697 and the fax number is 703-746-3098. The examiner can normally be reached on 7:45 to 4:45.

If attempts to reach the examiner by telephone are unsuccessful, the examiner's supervisor, Johann Richter can be reached on 571 272 0646. The fax phone number for the organization where this application or proceeding is assigned is 571-273-8300 ~~703-872-9306~~. Information regarding the status of an application may be obtained from the Patent Application Information Retrieval (PAIR) system. Status information for published applications may be obtained from either Private PAIR or Public PAIR. Status information for unpublished applications is available through Private PAIR only. For more information about the PAIR system, see http://pair-direct.uspto.gov. Should you have questions on access to the Private PAIR system, contact the Electronic Business Center (EBC) at 866-217-9197 (toll-free).

Lansana Nyalley, Ph.D.
04/04/05

Shailendra Kumar.
Technology Center 1600
Primary Examiner.

Interview Summary	Application No. 10/829,193	Applicant(s) KIYOMORI ET AL.	
	Examiner Lansana Nyalley	Art Unit 1621	

All participants (applicant, applicant's representative, PTO personnel):

(1) Lansana Nyalley. (3)_____.

(2) _____. (4)_____.

Date of Interview: April -5-2005.

Type: a)☒ Telephonic b)☐ Video Conference
c)☐ Personal [copy given to: 1)☐ applicant 2)☐ applicant's representative]

Exhibit shown or demonstration conducted: d)☐ Yes e)☒ No.
If Yes, brief description: _____.

Claim(s) discussed: 6 and 7.

Identification of prior art discussed: NO.

Agreement with respect to the claims f)☒ was reached. g)☐ was not reached. h)☐ N/A.

Substance of Interview including description of the general nature of what was agreed to if an agreement was reached, or any other comments: Applicant sauthorized the Examiner to cancel claim 6 and to delete the letter"a" before the word "and" in claim 7.

(A fuller description, if necessary, and a copy of the amendments which the examiner agreed would render the claims allowable, if available, must be attached. Also, where no copy of the amendments that would render the claims allowable is available, a summary thereof must be attached.)

THE FORMAL WRITTEN REPLY TO THE LAST OFFICE ACTION MUST INCLUDE THE SUBSTANCE OF THE INTERVIEW. (See MPEP Section 713.04). If a reply to the last Office action has already been filed, APPLICANT IS GIVEN ONE MONTH FROM THIS INTERVIEW DATE, OR THE MAILING DATE OF THIS INTERVIEW SUMMARY FORM, WHICHEVER IS LATER, TO FILE A STATEMENT OF THE SUBSTANCE OF THE INTERVIEW. See Summary of Record of Interview requirements on reverse side or on attached sheet.

Examiner Note: You must sign this form unless it is an Attachment to a signed Office action.

Examiner's signature, if required

Summary of Record of Interview Requirements

Manual of Patent Examining Procedure (MPEP), Section 713.04, Substance of Interview Must be Made of Record
A complete written statement as to the substance of any face-to-face, video conference, or telephone interview with regard to an application must be made of record in the application whether or not an agreement with the examiner was reached at the interview.

Title 37 Code of Federal Regulations (CFR) § 1.133 Interviews
Paragraph (b)
In every instance where reconsideration is requested in view of an interview with an examiner, a complete written statement of the reasons presented at the interview as warranting favorable action must be filed by the applicant. An interview does not remove the necessity for reply to Office action as specified in §§ 1.111, 1.135. (35 U.S.C. 132)

37 CFR §1.2 Business to be transacted in writing.
All business with the Patent or Trademark Office should be transacted in writing. The personal attendance of applicants or their attorneys or agents at the Patent and Trademark Office is unnecessary. The action of the Patent and Trademark Office will be based exclusively on the written record in the Office. No attention will be paid to any alleged oral promise, stipulation, or understanding in relation to which there is disagreement or doubt.

The action of the Patent and Trademark Office cannot be based exclusively on the written record in the Office if that record is itself incomplete through the failure to record the substance of interviews.

It is the responsibility of the applicant or the attorney or agent to make the substance of an interview of record in the application file, unless the examiner indicates he or she will do so. It is the examiner's responsibility to see that such a record is made and to correct material inaccuracies which bear directly on the question of patentability.

Examiners must complete an Interview Summary Form for each interview held where a matter of substance has been discussed during the interview by checking the appropriate boxes and filling in the blanks. Discussions regarding only procedural matters, directed solely to restriction requirements for which interview recordation is otherwise provided for in Section 812.01 of the Manual of Patent Examining Procedure, or pointing out typographical errors or unreadable script in Office actions or the like, are excluded from the interview recordation procedures below. Where the substance of an interview is completely recorded in an Examiners Amendment, no separate Interview Summary Record is required.

The Interview Summary Form shall be given an appropriate Paper No., placed in the right hand portion of the file, and listed on the "Contents" section of the file wrapper. In a personal interview, a duplicate of the Form is given to the applicant (or attorney or agent) at the conclusion of the interview. In the case of a telephone or video-conference interview, the copy is mailed to the applicant's correspondence address either with or prior to the next official communication. If additional correspondence from the examiner is not likely before an allowance or if other circumstances dictate, the Form should be mailed promptly after the interview rather than with the next official communication.

The Form provides for recordation of the following information:

- Application Number (Series Code and Serial Number)
- Name of applicant
- Name of examiner
- Date of interview
- Type of interview (telephonic, video-conference, or personal)
- Name of participant(s) (applicant, attorney or agent, examiner, other PTO personnel, etc.)
- An indication whether or not an exhibit was shown or a demonstration conducted
- An identification of the specific prior art discussed
- An indication whether an agreement was reached and if so, a description of the general nature of the agreement (may be by attachment of a copy of amendments or claims agreed as being allowable). Note: Agreement as to allowability is tentative and does not restrict further action by the examiner to the contrary.
- The signature of the examiner who conducted the interview (if Form is not an attachment to a signed Office action)

It is desirable that the examiner orally remind the applicant of his or her obligation to record the substance of the interview of each case. It should be noted, however, that the Interview Summary Form will not normally be considered a complete and proper recordation of the interview unless it includes, or is supplemented by the applicant or the examiner to include, all of the applicable items required below concerning the substance of the interview.

A complete and proper recordation of the substance of any interview should include at least the following applicable items:

1) A brief description of the nature of any exhibit shown or any demonstration conducted,
2) an identification of the claims discussed,
3) an identification of the specific prior art discussed,
4) an identification of the principal proposed amendments of a substantive nature discussed, unless these are already described on the Interview Summary Form completed by the Examiner,
5) a brief identification of the general thrust of the principal arguments presented to the examiner,
   (The identification of arguments need not be lengthy or elaborate. A verbatim or highly detailed description of the arguments is not required. The identification of the arguments is sufficient if the general nature or thrust of the principal arguments made to the examiner can be understood in the context of the application file. Of course, the applicant may desire to emphasize and fully describe those arguments which he or she feels were or might be persuasive to the examiner.)
6) a general indication of any other pertinent matters discussed, and
7) if appropriate, the general results or outcome of the interview unless already described in the Interview Summary Form completed by the examiner.

Examiners are expected to carefully review the applicant's record of the substance of an interview. If the record is not complete and accurate, the examiner will give the applicant an extendable one month time period to correct the record.

Examiner to Check for Accuracy

If the claims are allowable for other reasons of record, the examiner should send a letter setting forth the examiner's version of the statement attributed to him or her. If the record is complete and accurate, the examiner should place the indication, "Interview Record OK" on the paper recording the substance of the interview along with the date and the examiner's initials.

Issue Classification	Application/Control No.	Applicant(s)/Patent under Reexamination	
	10/829,193	KIYOMORI ET AL.	
	Examiner	Art Unit	
	Lansana Nyalley	1621	

ISSUE CLASSIFICATION

ORIGINAL		CROSS REFERENCE(S)							
CLASS	SUBCLASS	CLASS	SUBCLASS (ONE SUBCLASS PER BLOCK)						
556	446	556	443						
INTERNATIONAL CLASSIFICATION									
C 0 7 F	7/18								
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	/								

LANSANA NYALLEY, 04-04-05
(Assistant Examiner) (Date)

Shailendra Kumar

SHAILENDRA KUMAR, 04-04-05
(Primary Examiner) (Date)

Total Claims Allowed: 14

O.G. Print Claim(s)	O.G. Print Fig.
1	-

[illegible] 4/1/05
(Legal Instruments Examiner) (Date)

☐ Claims renumbered in the same order as presented by applicant ☐ CPA ☐ T.D. ☐ R.1.47

Final	Original	Final	Original	Final	Original	Final	Original	Final	Original	Final	Original	Final	Original
1	1		31		61		91		121		151		181
2	2		32		62		92		122		152		182
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4	4		34		64		94		124		154		184
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Index of Claims

Application/Control No.	Applicant(s)/Patent under Reexamination
10/829,193	KIYOMORI ET AL.
Examiner	**Art Unit**
Lansana Nyalley	1621

√ Rejected	– (Through numeral) Cancelled	N Non-Elected	A Appeal
= Allowed	+ Restricted	I Interference	O Objected

Claim: Final	Claim: Original	Date: 4/4/05
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Claim: Final	Claim: Original	Date
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TITLE
PREPARATION OF SILYL KETENE ACETAL AND DISILYL KETENE ACETAL

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encoding, the decoding relationships will be as follows:

For y odd

$$y_1 = h_1 sy_1 + h_3 sy_0 + g_1 dy_1 + g_3 dy_0$$
$$y_3 = h_1 sy_2 + h_3 sy_1 + g_1 dy_2 + g_3 dy_1$$
$$\vdots \qquad (10)$$
$$y_{2n+1} = h_1 sy_{n+1} + h_3 sy_n + g_1 dy_{n+1} + g_3 dy_n$$

For y even

$$y_2 = h_2 sy_1 + h_4 sy_0 + g_2 dy_1 + g_4 dy_0$$
$$y_4 = h_2 sy_2 + h_4 sy_1 + g_2 dy_2 + g_4 dy_1$$
$$\vdots \qquad (11)$$
$$y_{2k} = h_2 sy_n + h_4 sy_{n-1} + g_2 dy_n + g_4 dy_{n-1}$$

The values mapped into the parent mode are accumulated and, as in the encoder, extra values at the end positions (e.g. at y_0 above) can be handled by "wrap-around", truncation, or other known means of handling end conditions.

Referring to Fig. 16, there is shown a flow diagram which, when taken together with the further flow diagrams referred to therein, is suitable for controlling the processor to implement an embodiment of the encoding apparatus and method in accordance with the invention. The block 1610 represents the generating of processed values by correlating wavelet-packets with the input signal samples. There are various ways in which this can be achieved, the routine of Fig. 17 describing an implementation of the present embodiment. The block 1620 represents the routine, described further in conjunction with Fig. 18, for computing the information costs of the processed values. As described further hereinbelow, there are various ways of computing the measure or cost of information contained in the processed values. In an illustrated embodiment, a thresholding procedure is utilized, and information cost is determined by the number of values which exceed a particular threshold. The block 1630 represents the routine of Fig. 19 for selection of an advantageous orthogonal

basis from among the processed values, the selection of the basis being dependent on the computed information costs. The block 1640 represents compiling encoded frames from the selected processed values which constitute the basis, such as for subsequent recovery after processing, storage, and/or transmission. This routine is described in conjunction with Fig. 20.

Referring to Fig. 17, there is shown a flow diagram of a routine for generating the processed values from the sampled signal, or signal portion, using a wavelet-packet basis. The block 1710 represents the reading in of the selected coefficients h_i, g_i. The block 1720 is then entered, this block representing the initializing of a level index at 0, the initializing of a node index at 1, and the initializing of a position index at 1. The sample data, considered as level 0 is then read in, as represented by the block 1725. The sample data may consist, for example, of 256 sequential samples of an acoustical signal to be compressed and transmitted. The level index is then incremented, as represented by block 1730, and the first level processed values are computed and stored in accordance with the relationships (4) and (5) set forth above (block 1735 and loops 1748 and 1760). For example, for the first position of the first node of level 1, s_1 will be computed. If the wavelet employed is representable by a filter having four coefficients, as in the example above, s_1 will be computed as the sum of h_1x_1, h_2x_2, h_3x_3, h_4x_4. If a wavelet of more vanishing moments is used, more coefficients will be employed. In general, it will be preferable to utilize a wavelet having several coefficients, greater than four, the above examples being set forth for ease of illustration.

In loop 1748, inquiry is made (diamond 1740) as to whether the last position of the current node has been reached. If not, the position index is incremented (block 1745), and the block 1735 is re-entered for computation of the next processed value of the current node and level. The loop 1748 is then continued until all processed values have been computed for the current node, whereupon inquiry is made (diamond 1750) as to whether the

last node of the current level has been reached. If not, the node index is incremented (block 1755) and the loop 1760 is continued until the processed values have been computed for all nodes of the current level. For the first level, there will be only two nodes, with the values thereof being computed in accordance with the relationships (4) and (5) set forth above.

When the inquiry of diamond 1750 is answered in the affirmative, diamond 1770 is entered, and inquiry is made as to whether the last level has been processed. If not, the block 1730 is re-entered, to increment the level index, and the loop 1780 is continued until processed values have been obtained for the nodes at all levels of the tree.

Referring to Fig. 18, there is shown a flow diagram of the routine for computing the information cost of the nodes of the tree, so that an advantageous orthogonal basis can be selected. The block 1810 represents initializing the level index to the highest level (e.g., the last level in the illustration of Fig. 3). The node index and the position index are initialized at 1 (blocks 1815 and 1820). A node content count, which is used in the present embodiment to keep track of the number of processed values in a node that exceed a predetermined threshold, is initialized at zero, as represented by the block 1825. Inquiry is then made (diamond 1830) as to whether the value at the current position is less than a predetermined threshold value. If not, the node content count is incremented (block 1835), and the diamond 1840 is entered. If, however, the processed value at the current position is less than the threshold value, the diamond 1840 is entered directly. [At this point, the processed value could be set to zero prior to entry of diamond 1840 from the "yes" output branch of diamond 1830, but it is more efficient to handle this later.] Inquiry is then made (diamond 1840) as to whether the last position of the node has been reached. If not, the position index is incremented (block 1850), diamond 1830 is re-entered, and the loop 1855 is continued until all processed values of the current node have been considered. When this occurs, the node content count is stored for the current node (of the current level), as represented by the block 1860. Inquiry

is then made (diamond 1865) as to whether the last node of the level has been processed. If not, the block 1870 is entered, the node index is incremented, the block 1820 is re-entered, and the loop 1875 is continued until all nodes of the current level have been considered. Inquiry is then made (diamond 1880) as to whether the current level is the highest level. If so, there is no higher level against which comparison of parent-to-children node comparisons can be made. In such case, the level index is incremented (block 1885), block 1815 is re-entered, and the procedure just described is repeated to obtain and store node content counts for each node of the next-to-highest level. When this has been done, the inquiry of diamond 1880 will be answered in the negative, and the next routine (Fig. 19) will be operative to compare the level (which has just been processed to compute information cost of each node) with the children nodes of the previously processed higher level.

In particular, the level index is initialized (block 1905) to the highest level less one, and all nodes on the highest level are marked "kept". The node index is initialized (block 1910) and the node content count of the current node of the current level is compared (block 1920) to the sum of the node content counts of the two nodes which are children of the current node. [For example, if the current node is N_i and the current level is L_j, then the count for the current node is compared to the sum of the counts for nodes N_{2i-1} and N_{2i} of level L_{j+1}.] If the comparison shows that the parent has an equal or lower count, the parent is marked "kept", and the two children nodes are marked "not kept" (as represented by the block 1930). Conversely, if the comparison shows that the sum of two children nodes has a lower count than the parent node, each of the children nodes keeps its current mark, and the current parent node is marked "not kept" (as represented by the block 1940). In the case where the children nodes are preferred, the sum of the counts of the children nodes are attributed to the parent node (block 1945). By so doing, the lowest count will be used for subsequent comparisons as ancestors are examined. The attribution of the count to the parent node will not be problematic, since only

"kept" nodes will be considered in the next stage of processing. Inquiry is then made (diamond 1950) as to whether the last node of the current level has been reached. If not, the node index is incremented (block 1955), block 1920 is re-entered, and the loop 1958 is continued until all nodes at the current level have been considered. Inquiry is then made (diamond 1960) as to whether the current level is level 1. If not, the level index is decremented (block 1965), block 1815 (Fig. 18) is re-entered, and the loop 1970 continues until all levels have been considered. At this point, the nodes which define the basis to be used have been marked "kept" [possibly together with some of their descendent nodes], and correspond, for example, to the shaded nodes of the Fig. 4 illustrations.

Referring to Fig. 20, there is shown a flow diagram of a routine for generating output encoded words which, in the present embodiment, are collected in a frame which represents the encoded form of the data x_1, x_2, x_3 ... x_n. For example, for an acoustical signal, the frame may represent a particular number of acoustical samples, for example 256 samples. As a further example, for a video signal, the frame may represent a frame of video, portion thereof, or transformed frequency components thereof. The number of encoded words in a frame will generally vary as the information being encoded varies, and will also generally depend upon the level of the threshold employed, if a threshold is employed as in the present embodiment. Fig. 20 illustrates an embodiment of a routine for generating a frame of words for the basis that was selected using the routines of Fig.s 18 and 19. A tree location index will be calculated which points to nodes in the tree in depth-first order (or so-called "pre-order"), as is well known in the art. The tree location index is initialized to 1 at level 0, node 1 (block 2010). Inquiry is made (diamond 2015) as to whether the node at that tree location is market "kept", and, if not, diamond 2020 is entered directly, and inquiry is made as to whether the entire tree has been examined, as indicated by the tree location index. If the entire tree has been searched, block 2080 is entered and a "frame complete" indication can be generated. If not, then loop

2011 is continued until a node marked "kept" is encountered, or until the entire tree has been searched. If a node marked "kept" is encountered, block 2030 is entered, and the tree location index of this "kept" node is recorded in memory; suppose for example that it is called "X". The position index in the node is initialized (block 2035). Inquiry is then made (diamond 2040) as to whether the value at the current position is above the predetermined threshold. If not, diamond 2055 is entered directly, and no word is generated for the value at the current position. If the value is above the threshold, block 2045 is entered, this block representing the generation of a word which includes the current level, node, and position, and the value at the position. The block 2050 is then entered, this block representing the loading of the just generated word into an output register. Inquiry is then made (diamond 2055) as to whether the last position in the current node has been reached. If not, the position is incremented (block 2060), diamond 2040 is re-entered, and the loop 2061 is continued until all positions in the node "X" have been considered. It will be understood that various formats can be used to represent the words. For example, a specific number of bits can be used for each of the level, node, position, and value. Alternatively, words could be of different length, e.g. depending on information content or entropy, with suitable delineation between words, as is known in the art. Also, if desired, all words in a particular node could be encoded with a single indication of level and node, with individual indications of position-value pairs. Inquiry is next made (diamond 2070), as to whether the last node location in the tree in depth-first order has been reached. If not, the tree location index is incremented (block 2070), and inquiry is made as to whether the new node is a descendant of "X", by a comparison of depth-first indices well known in the art. When this is the case, diamond 2065 is re-entered, and the loop 2071 is continued until the first node which is not a descendant of "X" is encountered, or until there are no more nodes. When a first non-descendant of "X" is encountered, diamond 2015 is re-entered and the loop 2081 is continued until all nodes which

are both marked "kept" and have no ancestors marked "kept" have contributed to the frame. Such nodes contain a complete orthogonal group of wavelet-packet correlations (see also Appendix I, II, and V). When either loop 2011 or the loop 2071 terminates by exhaustion of the nodes, block 2080 is entered and a "frame complete" indication can be generated. If desired, the frame can then be read out of the encoder register. However, it will be understood that the encoder register can serve as a buffer from which words can be read-out synchronously or asynchronously, depending on the application.

Referring to Fig. 21, there is shown a flow diagram of the decoder routine for processing frames of words and reconstructing the orthogonal basis indicated by the words of a frame. The block 2110 represents the reading in of the next frame. In the described embodiment, it is assumed that the frames of words are read into a buffer (e.g. associated with decoder processor subsystem 170 of Fig. 1), and the individual words processed sequentially by placement into appropriate addresses (which can be visualized as the selected basis nodes of a tree - as in Fig. 4), from which reconstruction is implemented in the manner to be described. However, it will be understood that individual words can be received synchronously or asynchronously, or could be output in parallel into respective tree nodes, if desired. Also, as was the case with the encoder, parallel processing or network processing could be employed to implement reconstruction, consistent with the principles hereof. In the routine of Fig. 21, the next word of the frame is read (block 2115), and a determination is made as to whether the node and level of the word is occurring for the first time (diamond 2117). If so the node (and its level) is added to the list of nodes (block 2118). The value indicated in the word is stored (block 2120) at a memory location indicated by the level, node, and position specified in the word. It will be understood that memory need be allocated only for positions within the nodes designated by the read-in words. Inquiry is then made (diamond 2130) as to whether the last word of the frame has been reached. If not, the block 2115 is re-entered, and the loop 2135 is

continued until all words of the frame have been stored at appropriate locations. It will be understood that, if desired, the word locations (level, node, and position) could alternatively be stored, and the values subsequently recovered by pointing back to their original storage locations.

During the next portion of the decoder routine, as shown in Fig. 22, the values in the nodes on the list are utilized to implement reconstruction as in the diagram of Fig.s 15A and 15B, with parent nodes being reconstructed from children nodes until the level zero information has been reconstructed. During this procedure, when a parent node is reconstructed from its children nodes, the parent node is added to the list of nodes, so that it will be used for subsequent reconstruction. This part of the procedure begins by initializing to the first node on the list (block 2210). Next, the block 2215 represents going to the memory location of the node and initializing to the first position in the node. Inquiry is then made (diamond 2220) as to whether there is a non-zero value at the position. If not, diamond 2240 is entered directly. If so, the value at the position is mapped into the positions of the parent node, with accumulation, as described above in conjunction with relationships (10) and (11). Inquiry is then made (diamond 2240) as to whether the last position of the node has been reached. If not, the next position in the node is considered (block 2245), diamond 2220 is re-entered, and the loop 2250 continues until all positions in the node have been considered. It will be understood that, if desired, a marker or vector can be used to indicate strings of blank positions in a node, or to point only to occupied positions, so that a relatively sparse node will be efficiently processed. In this regard, reference can be made to the abovereferenced U.S. Patent Application Serial No. 525,974. When the last position of the node has been considered, the node is removed from the list of nodes, as represented by block 2255, and inquiry is made (diamond 2260) as to whether the parent node is at level 0. If so, diamond 2270 is entered directly. If, however, the parent node is not at level 0, the parent node is added to the list of nodes (block 2265). Inquiry is then made

(diamond 2270) as to whether the last node on the list has been reached. If not, the next node on the list is considered (block 2275), block 2215 is re-entered, and the loop 2280 is continued until processing is complete and the reconstructed values have been obtained. The decoder output values can then be read out (block 2290).

It will be understood that similar techniques can be employed at higher dimensions and in other forms (see e.g. Appendix V). More complicated tree structures, such as where a node has more than two children (e.g. Appendices II and V) can also be utilized.

The invention has been described with reference to particular preferred embodiments, but variations within the spirit and scope of the invention will occur to those skilled in the art. For example, it will be recognized that the wavelet upon which the wavelet-packets are based can be changed as different parts of a signal are processed. Also, the samples can be processed as sliding windows instead of segments.

I-1

Appendix I

Construction of Wavelet-Packets

We introduce a new class of orthonormal bases of $L^2(\mathbf{R}^n)$ by constructing a "library" of modulated wave forms out of which various bases can be extracted. In particular, the wavelet basis, the walsh functions, and rapidly oscillating wave packet bases are obtained.

We'll use the notation and terminology of [D], whose results we shall assume.

§1. We are given an exact quadrature mirror filter $h(n)$ satisfying the conditions of Theorem (3.6) in [D], p. 964, i.e.

$$\sum h(n-2k)h(n-2\ell) = \delta_{k,\ell}\,, \ \sum h(n) = \sqrt{2}.$$

We let $g_k = h_{k+1}(-1)^k$ and define the operations F_i on $\ell^2(\mathbf{Z})$ into "$\ell^2(2\mathbf{Z})$"

$$F_0\{s_k\}(i) = 2\sum s_k h_{k-2i} \tag{1.0}$$
$$F_1\{s_k\}(i) = 2\sum s_k g_{k-2i}.$$

The map $\mathbf{F}(s_k) = F_0(s_k) \oplus F_1(s_k) \in \ell^2(2\mathbf{Z}) \oplus \ell^2(2\mathbf{Z})$ is orthogonal and

$$F_0^* F_0 + F_1^* F_1 = I \tag{1.1}$$

We now define the following sequence of functions.

$$\begin{cases} W_{2n}(x) = \sqrt{2}\sum h_k W_n(2x-k) \\ W_{2n+1}(x) = \sqrt{2}\sum g_k W_n(2x-k) \end{cases} \tag{1.2}$$

Clearly the function $W_0(x)$ can be identified with the function φ in [D] and W_1 with the function ψ.

Let us define $m_0(\xi) = \frac{1}{\sqrt{2}}\sum h_k e^{-ik\xi}$ and

$$m_1(\xi) = -e^{i\xi}\bar{m}_0(\xi+\pi) = \frac{1}{\sqrt{2}}\sum g_k e^{ik\xi}$$

I-2

Remark: The quadrature mirror condition on the operation $\mathbf{F} = (F_0, F_1)$ is equivalent to the unitarity of the matrix

$$\mathcal{M} = \begin{bmatrix} m_0(\xi), m_1(\xi) \\ m_0(\xi+\pi), m_1(\xi+\pi) \end{bmatrix}$$

Taking Fourier transform of (1.2) when $n = 0$ we get

$$\hat{W}_0(\xi) = m_0(\xi/2)\hat{W}_0(\xi/2)$$

i.e.,

$$\hat{W}_0(\xi) = \prod_{j=1}^{\infty} m_0(\xi/2^j)$$

and

$$\hat{W}_1(\xi) = m_1(\xi/2)\hat{W}_0(\xi/2) = m_1(\xi/2)m_0(\xi/4)m_0(\xi/2^3)\cdots$$

More generally, the relations (1.2) are equivalent to

(1.3) $$\hat{W}_n(\xi) = \prod_{j=1}^{\infty} m_{\varepsilon_j}(\xi/2^j)$$

and $n = \sum_{j=1}^{\infty} \varepsilon_j 2^{j-1}$ ($\varepsilon_j = 0$ or 1).

We can rewrite (1.1) as follows.

(1.4) $$\begin{aligned} W_{2n}(x-\ell) &= \sqrt{2}\sum h_{j-2\ell}W_n(2x-j) = F_0\{W_n(2x-j)\}(\ell) \\ W_{2n+1}(x-\ell) &= \sqrt{2}\sum g_{j-2\ell}W_n(2x-j) = F_1\{W_n(2x-j)\}(\ell) \end{aligned}$$

where $W_n(2x-j)$ is viewed as a sequence in j for (x, n) fixed. Using (1.1) we find:

(1.5) $$\boxed{W_n(x-j) = \sqrt{2}\sum_i h_{j-2i}W_{2n}\left(\frac{x}{2}-i\right) + g_{j-2i}W_{2n+1}\left(\frac{x}{2}-i\right)}$$

In the case $n = 0$ we obtain:

(1.6) $$W_0(x-k) = \sqrt{2}\sum h_{k-2i}W_0\left(\frac{x}{2}-i\right) + g_{k-2i}W_1\left(\frac{x}{2}-i\right)$$

from which we deduce the usual decomposition of a function f in the space Ω_0 (V_0 in [D]) i.e., a function f of the form

$$\begin{aligned} f(x) &= \sum s_k^0 W_0(x-k) \\ &= \sqrt{2}\sum(\sum s_k^0 h_{k-2i})W_0\left(\frac{x}{2}-i\right) + \sqrt{2}\sum(\sum s_k^0 g_{k-2i})W_1\left(\frac{x}{2}-i\right) \\ &= \sum \frac{1}{\sqrt{2}} F_0(s_k^0)(i) W_0\left(\frac{x}{2}-i\right) + \sum \frac{1}{\sqrt{2}} F_1(s_k^0)(i) W_1\left(\frac{x}{2}-i\right) \end{aligned}$$

More generally, if we define

$$\Omega_n = \{f : f = \sum \omega_k W_n(x-k)\}. \tag{1.7}$$

We find

$$f(x) = \sum \frac{1}{\sqrt{2}} F_0(\omega_k)(i) W_{2n}\left(\frac{x}{2}-i\right) + \sum \frac{1}{\sqrt{2}} F_1(\omega_k)(i) W_{2n+1}\left(\frac{x}{2}-i\right) \tag{1.8}$$

or

$$\sqrt{2}f(2x) = h + g \qquad h \in \Omega_{2n}\ g \in \Omega_{2n+1}$$

We now prove

Theorem (1.1). *The functions $W_n(x-k)$ form an orthonormal basis of $L^2(\mathbb{R})$.*

Proof. We proceed by induction on n, assuming that $W_n(x-k)$ form an orthonormal set of functions and, proving that, $W_{2n}(x-k), W_{2n+1}(x-k)$ form an *o.n* set.

By assumption $\|\sqrt{2}f(2x)\|_2^2 = \sum \omega_k^2$ if $f \in \Omega_n$ from the quadrature mirror condition on (F_0, F_1) we get

$$\sum \omega_k^2 = \sum F_0(\omega_k)(i)^2 + F_1(\omega_k)(i)^2.$$

Since $F_0(\omega_k)(i) = \mu_i$, $F_0(\omega_k)(i) = \nu_i$ can be chosen as two arbitrary sequences of ℓ^2 (arising from $\omega = F_0^*\mu_i + F_1^*\nu_i$) it follows that

$$\int |\sum \mu_i W_{2n}(x-i) + \sum \nu_i W_{2n+1}(x-i)|^2 = \sum \mu_i^2 + \sum \nu_i^2$$

which is equivalent to $W_{2n}(x-i), W_{2n+1}(x-j)$ being an *o.n* set of functions.

Let us now define $\delta f = \sqrt{2}f(2x)$. Formula (1.8) shows that $\delta\Omega_n = \Omega_{2n} \oplus \Omega_{2n+1}$ as an orthogonal sum or,

$$\begin{aligned} \delta\Omega_0 - \Omega_0 &= \Omega_1 \\ \delta^2\Omega_0 - \delta\Omega_0 &= \delta\Omega_1 = \Omega_2 \oplus \Omega_3 \\ \delta^3\Omega_0 - \delta^2\Omega_0 &= \delta\Omega_2 \oplus \delta\Omega_3 = \Omega_4 \oplus \Omega_5 \oplus \Omega_6 \oplus \Omega_7 \text{ or} \\ \delta^k\Omega_0 - \delta^{k-1}\Omega_0 &= \Omega_{2^{k-1}} \oplus \Omega_{2^{k-1}+1} \cdots \oplus \Omega_{2^k-1} \end{aligned} \tag{1.9}$$

and

$$\delta^k \Omega_0 = \Omega_0 \oplus \Omega_1 \oplus \cdots \oplus \Omega_{2^k-1}$$

More generally, we let $\mathcal{W}_k = \delta^{k+1}\Omega_0 - \delta^k\Omega_0 = \delta^k\Omega_1 = \delta^k\mathcal{W}_1$. Therefore we have

Proposition (1.1).

$$\mathcal{W}_k = \delta^k W_1 = \Omega_{2^k} \oplus \Omega_{2^k+1} \oplus \cdots \oplus \Omega_{2^{k+1}-1}.$$

Alternatively, the functions

$$W_n(x-j) \qquad j \in \mathbf{Z}\ 2^k \leq n < 2^{k+1}$$

form an orthonormal basis of $\mathcal{W}_k$.

Since the spaces $\mathcal{W}_k$ are mutually orthogonal and span $L^2(\mathbf{R})$ see [D], it follows that $W_n(x-j)$ are complete.

§2. Orthonormal bases extracted out of the "library" $2^{j/2}W_n(2^jx-k)$.

We start by observing that $2^{j/2}W_1(2^jx-k)$ form a basis of $\mathcal{W}_j$ as we vary k and a basis of $L^2(\mathbf{R})$ as j,k vary. This is the wavelet basis constructed in [D]. The following simple generalization is useful to obtain a better localization in frequency space.

Theorem (2.1). *The functions*

$$2^{j/2}W_n(2^jx-k) \qquad j = 0, \pm 1, \ldots, k = 0, \pm 1, \pm 2$$

$2^\ell \leq n < 2^{\ell+1}$ *for fixed ℓ form an orthonormal basis of $L^2(\mathbf{R})$ $\ell = 0, 1, 2, \ldots$.*

Remark: This is a wavelet basis with dyadic dilations and 2^ℓ fundamental wavelets.

Proof. We have seen, in Proposition (1.1), that $W_n(x-k)$ $2^\ell \leq n < 2^{\ell+1} k \in \mathbf{Z}$ form an *o.n* basis of $\mathcal{W}_\ell$, from which we deduce that $2^{j/2}W_n(2^jx-k)$ form an *o.n.* basis of $\mathcal{W}_{\ell+j}$ spanning $L^2(\mathbf{R})$ for $j \in \mathbf{Z}$ $k \in \mathbf{Z}$.

In the next example we vary j and n simultaneously to obtain a basis whose number of oscillation is inversely proportional to the length of its support.

Theorem (2.2). *Let $\ell(n) = [\log_2 n]$ i.e., $2^{\ell(n)} \leq n < 2^{\ell+1}$ then*

$$W_n(2^{\ell(n)}x-k)2^{\ell(n)/2},\ W_n(2^{\ell(n)+1}x-k)2^{\frac{\ell(n)+1}{2}}$$

form an o.n basis of $L^2(\mathbf{R})$.

Proof. Fix $\ell(n) = \ell$, and consider n with $2^\ell \leq n < 2^{\ell+1}$. as seen in the proof of Theorem (1.2)

$$W_n(2^\ell x-k)2^{\ell/2} \quad \text{form an } o.n \text{ basis of } \mathcal{W}_{2\ell}$$

and

$$W_n(2^{\ell+1}x-k)2^{(\ell+1)/2} \quad \text{form an } o.n \text{ basis of } \mathcal{W}_{2\ell+1}$$

since $L^2 = \sum \oplus \mathcal{W}_{2\ell} \oplus \sum \oplus \mathcal{W}_{2\ell+1}$, we have a complete basis.

These can be generalized as follows.

Theorem (2.3). *Let a collection $\{\ell, n\}$ be given such that the dyadic intervals $I_{\ell,n} = [2^\ell n, 2^\ell(n+1))$ form a disjoint covering of $(0,\infty)$, then $2^{\ell/2}W_n(2^\ell x-k)$ form a complete orthonormal basis of $L^2(\mathbf{R})$.*

This theorem becomes obvious in the following case. Let

$$m_0(\xi) = \begin{cases} 1 & |\xi| < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le |\xi| < \pi \end{cases}$$

a periodic function of period 2π, and $m_1(\xi) = 1 - m_0(\xi)$. Let

$$\hat{\omega}_n(\xi) = \prod_{j=1}^{\infty} m_{\varepsilon_j}(\xi/2^j) \quad n = \sum^{\infty} \varepsilon_j 2^{j-1}$$

then

$$\hat{\omega}_n(\xi) = \begin{cases} 1 & n \le |\xi/\pi| < n+1 \\ 0 & \text{elsewhere} \end{cases}$$

and the orthonormal basis $\omega_n(x-k)$ in Fourier space is

$$e^{ik\xi}\hat{\omega}_n(\xi)$$

which is the simplest variation on a "windowed" (2 windows) Fourier transform. Theorem (1.4) is obvious in this case. This theorem is also easy to understand from the point of view of subband coding as we shall see.

§3. Subband coding and expansions in terms of W_n

We assume, given a function which. on the scale 2^{-N}, is well approximated as

(3.1)
$$f(x) = \sum s_k^0 W_0(2^N x - k)2^{N/2}$$

as seen in (1.8)

(3.2)
$$f(x) = \frac{1}{\sqrt{2}}\sum\left\{F_0(s_k^0)(i)W_0\left(\frac{x2^N}{2}-i\right) + F_i(s_k^0)(i)W_1\left(\frac{x2^N}{2}-i\right)\right\}2^{N/2}$$
$$= \left\{\frac{1}{\sqrt{2}}f_0\left(\frac{x}{2}2^N\right) + \frac{1}{\sqrt{2}}f_1\left(\frac{x}{2}2^N\right)\right\}2^{N/2} \text{ with } f_0 \in \Omega_0 f_1 \in \Omega_1$$

The coefficients of f_0 are given by $F_0(s^0)$.
The coefficients of f_1 are given by $F_1(s^0)$.
Continuing an application of (1.8) gives

$$= \frac{2^{N/2}}{2}\left\{f_{00}\left(\frac{x}{4}2^N\right) + f_{10}\left(\frac{x}{4}2^N\right) + f_{01}\left(\frac{x}{4}2^N\right) + f_{11}\left(x22^N\right)\right\}$$

where f_{00}, f_{10} are obtained by decomposing f_0
and f_{01}, f_{11} are obtained by decomposing f_1
$f_{00} \in \Omega_0, f_{10} \in \Omega_1, f_{01} \in \Omega_2, f_{11} \in \Omega_3$. If we continue this decomposition and observe that the binary tree corresponds to the realization of n as $n = \sum_1 \varepsilon_j 2^{j-1}$ and that after ℓ iterations we get

$$f(x) = 2^{(N-\ell)/2} \sum_{k=0}^{2^\ell - 1} f_n(x2^{N-\ell}) \text{ with } f_n \in \Omega_n. \tag{3.3}$$

and $f_n(x) = \sum_k \langle f, W_n(2^{N-\ell}x - k)\rangle W_n(2^{N-\ell}x - k)2^{N-\ell}$ with

$$\begin{aligned} 2\frac{N-\ell}{2}\langle f, W_n(2^{N-\ell} - k)\rangle &= F_{\varepsilon_1}F_{\varepsilon_2}\ldots F_{\varepsilon_\ell}\{s_k^0\} \\ n &= \sum \varepsilon_j 2^{j-1}. \end{aligned} \tag{3.4}$$

We therefore obtain a fast $2^N N$ algorithm to calculate all coefficients for "all functions in our library" for scales $-N \le j \le 0$. The procedure is analogous to subband coding.

§4. Higher dimensional libraries and bases

We define the higher dimensional wavelet basis as follows: Let

$\mathcal{W}_1 =$ span$\{W_0(x_1-k_1)W_1(x_2-k_2), W_1(x_1-k_1)W_0(x_2-k_2), W_1(x_1-k_1)W_1(x_2-k_2)\}$

$\mathcal{W}_k = \delta^k W_1$ where $\delta f(x_1 x_2) = 2f(2x_1, 2x_2)$. Clearly

$$L^2(\mathbf{R}^2) = \sum \mathcal{W}_k.$$

The basic 2-dimensional "library" consists of all functions obtained as tensor products of the one dimensional library i.e.,

$$2^{\frac{1}{2}(j_1+j_2)}W_{n_1}(2^{j_1}x_1 - k_1)W_{n_2}(2^{j_2}x_2 - k_2).$$

We will restrict our attention to the sublibraries obtained by dilating both variables by the same dilations although the case $j_2 = rj_1$ r fixed is of independent interest.

The two dimensional basis corresponding to Theorem (2.1) is given in

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Theorem (4.1). *Fix ℓ, then for $(k_1, k_2) \in \mathbb{Z}^2$ $j \in \mathbb{Z}$ $2^\ell \le n < 2^{\ell+1}$,*

$$2^{j/2} W_n(2^j x_1 - k_1) 2^{\frac{j+\ell}{2}} W_0(2^{\ell+j} x_2 - k_2)$$
$$2^{\frac{(j+\ell)}{2}} W_0(2^{\ell+j} x_1 - k_1) 2^{j/2} W_n(2^j x_2 - k_2)$$
$$2^j W_{n_1}(2^j x_1 - k_1) W_{n_2}(2^j x_2 - k_2) \quad 2^\ell \le n_i < 2^{\ell+1}$$

form an orthonormal basis of L^2.

Theorem (4.2). *Let $\langle n \rangle = \langle n_1, n_2 \rangle = 2^{\max(\ell(n_1)\ell(n_2))}$ where $\ell(n_1) = [\log_2 n_1]$ $n_1 \ge 0$ $n_2 \ge 0$ then for $(k_1, k_2) \in \mathbb{Z}^2$,*

$$\langle n \rangle \cdot W_{n_1}(\langle n \rangle x_1 - k_1) W_{n_2}(\langle n \rangle x_2 - k_2)$$
$$\langle 2n \rangle W_{n_1}(2\langle n \rangle x_1 - k_1) W_{n_2}(2\langle n \rangle x_2 - k_2)$$

form an orthonormal basis (wavelet packet basis of $L^2(\mathbb{R}^2)$).

Proof. Assume $\ell = \max(\ell(n_1)\ell(n_2)) = \ell(n_2)$, i.e. $n_1 \le n_2$. Then

$$2^\ell W_{n_1}(2^\ell x_1 - k_1) W_{n_2}(2^\ell x_2 - k_2)$$

for

$$0 \le n_1 < 2^\ell \qquad 2^\ell \le n_2 < 2^{\ell+1}$$

span (using 1.9) Proposition (1.1) $\delta^{2\ell}\Omega_{0,x_1} \otimes \delta^{2\ell}\Omega_{1,x_2}$ i.e.. the subspace spanned by $W_0(2^{2\ell}x_1 - k_1)W_1(2^{2\ell}x_2 - k_2)$. Consideration of the other 2 cases yields a total span of $\mathcal{W}_{2\ell}$, similarly we obtain $\mathcal{W}_{2\ell+1}$ and the theorem is proved.

Reference

[D] Ingrid Daubechies, *Orthonormal bases of compactly supported wavelets*, Communications on Pure and Applied Mathematics XLI (1988), 909–966.

Appendix II

Best-Adapted Wavelet-Packet Bases

Introduction. By using extra filters, it is possible to introduce fast wave packet transformations which decimate by arbitrary numbers. Such transformations generalize algorithms which decimate by 2. The method produces new libraries of orthonormal basis vectors. We introduce an algorithm for selecting a most efficient representation from the library, and prove that its complexity is $O(N)$ for a sequence of length N. We discuss some of the analytic properties and applications of such representations.

Aperiodic filters and bases in l^2. Consider first the construction of bases on l^2. Let p be a positive integer and introduce p absolutely summable sequences $f_0, \ldots, f_{p-1}$ satisfying the properties:

(1) For some $\epsilon > 0$, $\sum_m |f_i(m)||m|^\epsilon < \infty$,
(2) $\sum_m f_0(m) = 1$, for $i = 0, 1, \ldots, p-1$, and
(3) $\sum_m f_i(m) f_j(m + kp) = \delta_{i-j}\delta_k$, where δ is the Kronecker symbol.

To these sequences are associated p convolution operators $F_0, \ldots, F_{p-1}$ and their adjoints $F_0^*, \ldots, F_{p-1}^*$ defined by

$$F_i : l^2 \to l^2, \qquad F_i v(k) = \sum_m f_i(m + pk) v(m),$$
$$F_i^* : l^2 \to l^2, \qquad F_i^* v(m) = \sum_k f_i(m + pk) v(k).$$

These convolution operators will be called *filters* by analogy with quadrature mirror filters in the case $p = 2$. They have the following properties:

Lemma. *For $i, j = 0, 1, \ldots, p-1$,*

(1) $F_i F_j^* = 0$, *if $i \neq j$,*
(2) $F_i F_i^* = I$,
(3) $F_i^* F_i$ *is an orthogonal projection of l^2, and for $i \neq j$ the ranges of $F_i^* F_i$ and $F_j^* F_j$ are orthogonal, and*
(4) $F_0^* F_0 + \cdots + F_{p-1}^* F_{p-1} = I$.

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Proof. Properties (1) and (2) follow by interchanging the order of integration:

$$\begin{aligned} F_iF_j^*v(k') &= \sum_m \sum_k f_i(m+pk')f_j(m+pk)v(k) \\ &= \sum_k \left(\sum_{m'} f_i(m')f_j(m'+p[k-k']) \right) v(k) \\ &= \sum_k \delta_{i-j}\delta_{k-k'}v(k) \\ &= \begin{cases} v(k'), & \text{if } i=j, \\ 0, & \text{if } i \neq j, \end{cases} \end{aligned}$$

by the orthogonality properties of f_i.

Property (3) follows from (1) and (2): $F_i^*F_iF_i^*F_i = F_i^*F_i$, and $F_i^*F_iF_j^*F_j = 0$. Orthogonality is easily shown by transposition.

To prove (4), let $m_j(\xi) = \sum_k f_j(k)e^{ik\xi}$ be the (bounded, Hölder continuous, periodic) function determined by the filter f_j, for $j = 0, \ldots, p-1$. Then $f_j(k) = \hat{m}_j(k)$ is a real number, and each $F_i^*F_i$ is unitarily equivalent to multiplication by $|m_i|^2$ on $L^2(-\pi, \pi)$.

Now Plancherel's theorem gives

$$\int_0^{2\pi} e^{ilp\xi} m_j(\xi)\bar{m}_{j'}(\xi)\, d\xi = \sum_k f_j(k)f_{j'}(k+lp) = \delta_{j-j'}\delta_l.$$

In particular, $|m_j|^2$ has integral 1, and the Fourier coefficient $(|m_j|^2)\hat{}(lp)$ vanishes if $l \neq 0$. This is equivalent to the average of $|m_j(\xi)|^2$ over $\{\xi, \xi+2\pi/p, \ldots, \xi+2\pi(p-1)/p\}$ being identically 1.

The same vanishing is true of the Fourier coefficients of the cross terms $m_j\bar{m}_{j'}$, and for those it also holds when $l = 0$. Thus, the average of $m_j(\xi)\bar{m}_{j'}(\xi)$ over $\{\xi, \xi + 2\pi/p, \ldots, \xi + 2\pi(p-1)/p\}$ vanishes identically. Hence, the conditions on the filters f_i are equivalent to the unitarity of the matrix

$$\begin{pmatrix} m_0(\xi) & m_0(\xi + \frac{2\pi}{p}) & \ldots & m_0(\xi + \frac{2\pi(p-1)}{p}) \\ \ldots & \ldots & \ldots & \ldots \\ m_{p-1}(\xi) & \ldots & \ldots & m_{p-1}(\xi + \frac{2\pi(p-1)}{p}) \end{pmatrix}$$

But then $\sum_{k=0}^{p-1} |m_k(\xi)|^2 = 1$ for all ξ. Thus $F_0^*F_0 + \cdots + F_{p-1}^*F_{p-1}$ is unitarily equivalent to multiplication by 1 in $L^2(-\pi, \pi)$, proving (4). □

With this proposition we can decompose l^2 into mutually orthogonal subspaces $W_0^1 \perp \cdots \perp W_{p-1}^1$, where $W_i^1 = F_i^*F_i(l^2)$ for $i = 0, \ldots, p-1$. The map F_i finds

the coordinates of a vector with respect to an orthonormal basis of W_i^1. Since each $F_iW_i^1 = F_i(l^2)$ is another copy of l^2, there is nothing to prevent us from reapplying the filter convolutions recursively. At the mth stage, we obtain $l^2 = W_0^m \perp \cdots \perp W_{p^m-1}^m$, where $W_n^m = F_{n_1}^* \ldots F_{n_m}^* F_{n_m} \ldots F_{n_1}(l^2)$ and $n_m \ldots n_1$ is the radix-p representation of n. The map $F_{n_m} \ldots F_{n_1}$ transforms into standard coordinates in W_n^m. For convenience, we will introduce the notations $\mathbf{F}_n^m = F_{n_m} \ldots F_{n_1}$, and $\mathbf{F}_n^{m*} = F_{n_1}^* \ldots F_{n_m}^*$.

The subspaces W_n^m form a p-ary tree. Every node W_n^m is a parent with p daughters $W_{pn}^{m+1}, \ldots, W_{pn+p-1}^{m+1}$. The root of the tree is the original space l^2, which we may label W_0^0 for consistency. Call the whole tree $\mathbf{W}$.

Now fix m and suppose w belongs to W_n^m, where $0 \leq n \leq p^m - 1$, and $\mathbf{F}_n^m w = e_k$ is the elementary sequence with 1 in the kth position and 0's elsewhere. The collection of all such w forms an orthonormal basis of l^2 with some remarkable properties. In particular, if $p = 2$ and the filters F_0 and F_1 are taken as low-pass and high-pass quadrature mirror filters, respectively, then the spaces $W_0^m, \ldots W_{2^m-1}^m$ are all the subbands at level m. These have been used for a long time in digital signal processing and compression. In an earlier paper we described experiments with an algorithm for choosing m so as to reduce the bit rate of digitized acoustic signal transmission. This produced good signal quality at rather low bit rates.

The tree contains other orthogonal bases of W_0^0. In fact, it forms a library of bases which may be adapted to classes of functions. The tree structure allows the library to be searched efficiently for the extremum of any additive functional.

To every node in $\mathbf{W}$ we associate the subtree of all its descendants. Define a *graph* to be any subset of the nodes of $\mathbf{W}$ with the property that the union of the associated subtrees is disjoint and contains a complete level $W_0^m, \ldots, W_{p^m-1}^m$ for some m. The singleton $\{W_0^0\}$ is a graph, for example, with $m = 0$. The following may be called the graph theorem.

Theorem. *Every graph corresponds to a decomposition of l^2 into a finite direct sum.*

Proof. Every graph is a finite set, of cardinality no more than p^m for the m in the definition. Fix a graph, and suppose that $W_{n_1}^{m_1}$ and $W_{n_2}^{m_2}$ are subspaces corresponding to two nodes. Without loss, suppose that $m_1 \leq m_2$. Then $W_{n_2}^{m_2}$ is contained in a subspace $W_n^{m_1}$ for some $n \neq n_1$. Since the subspaces at a given level are orthogonal, we conclude that $W_{n_2}^{m_2} \perp W_{n_1}^{m_1}$.

To show that the decomposition is complete, observe that a node contains the sum of its daughters. By induction, it contains the sum of all of the nodes in its subtree. Hence a graph contains the sum of all the subspaces at some level m. But this sum is

all of l^2. $\square$

Theorem. *Graphs are in one-to-one correspondence with finite disjoint covers of* $[0,1)$ *by p-adic intervals* $I_n^m = p^{-m}[n, n+1)$, $n = 0, 1, \ldots, p^m - 1$.

Proof. The correspondence is evidently $W_n^m \leftrightarrow I_n^m$. The subtree associated to W_n^m corresponds to all p-adic subintervals of I_n^m. The details are left to the reader. $\square$

Analytic properties of graphs: continuous wave packets. Each filter F_j (and its adjoint F_j^*) maps the class of exponentially decreasing sequences to itself. Likewise, the projections $\mathbf{F}_n^{m*}\mathbf{F}_n^m$ preserve that class. In practice, we shall consider only finite sequences in l^2. For actual computations the filters must be finitely supported as well. Convolution with such filters preserves the property of finite support. Let the support width of the filters be r, and let z_m be the maximum width of any vector of the form $F_{j_1}^* \ldots F_{j_m}^*(e_k)$. Then $z_0 = 1$ and $z_{m+1} = pz_m + r - p$. By induction, we see that $z_m = p^m + (p^m - 1)(r - p)$.

Coifman and Meyer [CM] have observed that the basis elements $\mathbf{F}_n^{m*}e_k$ are related to wave packets over $\mathbf{R}$. A slightly generalized paraphrase of their construction follows. Many of the basic facts used were proved by Daubechies in [D].

Let w be a function defined by $\hat{w}(\xi) = \prod_{j=1}^{\infty} m_0(\xi/p^j)$, where m_0 is the analytic function defined by F_0, as above. Then w has mass 1, decreases rapidly, and is Hölder continuous, as proved in [D]. If m_0 is a trigonometric polynomial of degree r, then w is supported in the interval $[-r, r]$. Arranging that w has r continuous derivatives requires m_0 with degree at most $O(r)$. See [D] for a discussion of the constant in this relation for $p = 2$. Put $w_0^0 = w$, and define the family of wave packets recursively by the formula $w_{pn+j}^{m+1}(t) = \sum_{-\infty}^{\infty} f_j(i) w_n^m(pt - i)$. This produces one function w_n^m for each pair (m, n), where $m = 0, 1, \ldots$ and $n = 0, 1, \ldots, p^m - 1$.

We can renormalize the wave packets to a fixed scale p^L. Write

$$w_{n,m,k}^L(t) = p^{(L-m)/2} w_n^m(p^{L-m}t - k).$$

Then $w_{0,0,k}^L$ is a collection of orthonormal functions of mass 1, concentrated in intervals of size $O(p^{-L})$. This makes them suitable for sampling continuous functions. Let $x(t)$ be any continuous function, and put

$$s_0^0(k) = \langle x, w_{0,0,k}^L \rangle = \int_{-\infty}^{\infty} x(t) w_0^0(p^L t - k)\, dt.$$

We may use $s_0^0(k)$ as a representative value of $x(t)$ in the interval $I_k^L = p^{-L}[k, k+1)$. The closeness of the approximation to values of x depends, of course, on the smoothness

of x. Suppose that x is Hölder continuous with exponent ϵ. Then if t_0 is any point in I_k^L, we have

$$|x(t_0) - s_0^0(k)| = |\int_{I_k^L} (x(t_0) - x(t))\, w_0^0(p^L t - k)\, dt| = O(p^{-\epsilon L}).$$

We can also take advantage of differentiability of x if we construct w_0^0 with vanishing moments. Given d vanishing moments and d derivatives of x, the approximation improves to $|x(t_0) - s_0^0(k)| = O(p^{-dL})$.

The map $x \mapsto s_0^0$ sends $L^2(\mathbf{R})$ to l^2, and pulls back the orthonormal bases of l^2 constructed in the last section. To see this, define $s_n^m(k) = \langle x, w_{n,m,k}^L\rangle$. By interchanging the order of recurrence relation and inner product, we obtain the formula $s_n^m = \mathbf{F}_n^m s_0^0$. Thus, the coordinates $s_n^m(k)$ are coefficients with respect to an orthonormal basis of W_n^m.

The resulting subspaces of $L^2(\mathbf{R})$ form a finer type of multiresolution decomposition than that of Mallat [Ma]. The coordinates $s_n^m(k)$ are rapidly computable. They contain a mixture of location and frequency information about x.

Ordering the basis elements. The parameters n, m, k, L in $w_{n,m,k}^L$ have a natural interpretation as frequency, scale, position, and resolution, respectively. However, n is not monotonic with frequency, because our construction yields wave packets in the so-called Paley (natural, or p-adic) ordering. The following results show how to permute $n \mapsto n'$ into a frequency-based ordering.

Theorem. *We can choose rapidly decreasing filters $F_0, \ldots, F_p - 1$ such that $w_{n,m,k}^L$ is concentrated near the interval I_k^{L-m}, and $\hat{w}_{n,m,k}^L$ is concentrated near the interval $I_{n'}^m$, where $n \mapsto n'$ is a permutation of the integers.*

Proof. The first part is evident. For any family of exponentially decreasing filters, w_0^0 decreases exponentially away from $[0,1)$. $w_{0,m,k}^L$ is its dilate and translate to the interval I_k^{L-m}. Likewise, $w_{n,m,k}^L$ has the same concentration as $w_{0,m,k}^L$, since all the filters F_i are uniformly exponentially decreasing.

The second part follows from the Fourier transform of the recurrence relation:

$$\hat{w}_{pn+j}^{m+1}(\xi) = \left(p^{-1} \sum_k f_j(k) e^{-ix\xi/p}\right) \hat{w}_n^m(\xi/p) = p^{-1} m_j(\xi/p) \hat{w}_n^m(\xi/p),$$

where m_j is the multiplier defined above. Recall that $\sum_{j=0}^{p-1} |m_j(\xi)|^2 \equiv 1$ and that $m_0(0) = 1$. Thus, the periodic functions $|m_j|^2$ form a partition of unity into p functions. with 0 being in the support of m_0 alone.

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Now suppose for simplicity that we have chosen filters in such a way that $|m_j(\xi)| = \sum_{k=-\infty}^{\infty} \chi_{\pm\frac{\pi}{p}[j,j+1)}(\xi - 2\pi k)$. Such m_j may be approximated in $L^2(-\pi,\pi)$ as closely as we like by multipliers arising from exponentially decreasing filters. In this simple case, it is immediate that $\hat{w}_0^0(\xi) = m_0(\xi/p)|_{(-\pi,\pi)}$ is the characteristic function of $(-\pi,\pi)$, so that $\hat{w}_{0,0,0}^L$ is the characteristic function of $(-\pi p^L, \pi p^L)$. Likewise, $\hat{w}_{j,1,0}^L$ is the characteristic function of $\pi p^{L-1}(-j-1,-j] \cup \pi p^{L-1}[j,j+1)$. From the recurrence relation, we see that $\hat{w}_{n,m,0}^L$ will be the characteristic function of the union of the intervals $\pm\pi p^{L-m}[n', n'+1)$, where $n \mapsto n'$ is a permutation. These intervals cover $p^L(-\pi,\pi)$ as $n = 0,\ldots,p^m-1$. The permutation $n \mapsto n'$ is given by the recurrence relation

$$n' = n, \qquad \text{if } n = 0,\ldots,p-1; \qquad (np+j)' = \begin{cases} n'p + j, & \text{if } n' \text{ is even.} \\ n'p + (p-1) - j, & \text{if } n' \text{ is odd.} \end{cases}$$

Write n_j for the jth digit of n in radix p, numbering from the least significant. Set $n_m = 0$ if n has fewer than m digits. Then the recurrence relation implies that $n_j = \pi(n'_{j+1}, n'_j)$, where

$$\pi(x,y) = \begin{cases} y, & \text{if } x \text{ is even,} \\ p-1-y, & \text{if } x \text{ is odd..} \end{cases}$$

For each value of the first variable, π is a permutation of the set $\{0,\ldots,p-1\}$ in the second variable. Thus the map $n' \mapsto n$ and its inverse $n \mapsto n'$ are permutations of the integers. It is not hard to see that these are permutations of order 2 if p happens to be odd. Otherwise they have infinite order, as may be seen by considering an increasing sequence of integers n' all of which have only odd digits in radix p. □

Corollary. *With filters $F_0,\ldots,F_{p-1}$ chosen as above, we can modify the recurrence relation for $w_{n,m,k}^L$ such that $\hat{w}_{n,m,k}^L$ is concentrated near the interval I_n^m.*

Proof. Simply reorder the functions w_n^m by using the alternative recurrence relation:

$$w_{pn+j}^{m+1}(t) = \begin{cases} \sum_k f_j(k) w_n^m(pt-k), & \text{if } n \text{ is even,} \\ \sum_k f_{p-1-j}(k) w_n^m(pt-k), & \text{if } n \text{ is odd.} \end{cases}$$

Since we are enforcing $n = n'$ at each level m, we are composing with the permutation defined above. Of course, this algorithm has complexity identical to the original. □

Periodic filters and bases for $\mathbf{R}^d$. A sampled periodic function may be represented as a vector in $\mathbf{R}^d$ for some d. In this case let p be any factor of d. Introduce as filters a family of p vectors $\{\tilde{f}_i \in \mathbf{R}^d : i = 0,\ldots,p-1\}$. These are obviously summable.

Suppose in addition that they are orthogonal as periodic discrete functions, i.e., that $\sum_{m=1}^{d} \tilde{f}_i(m)\tilde{f}_j(m+kp \mod d) = \delta_{i-j}\delta_k$.

Let the associated convolution operators be $\{\tilde{F}_0, \ldots, \tilde{F}_{p-1}\}$, defined as above by

$$\tilde{F}_i : \mathbf{R}^d \to \mathbf{R}^{d/p}, \qquad \tilde{F}_i v(k) = \sum_{m=1}^{d} \tilde{f}_i(m+pk \mod d)v(m), \qquad \text{for } k = 1, 2, \ldots, d/p,$$

$$\tilde{F}_i^* : \mathbf{R}^{d/p} \to \mathbf{R}^d, \qquad \tilde{F}_i^* v(m) = \sum_{k=1}^{d/p} \tilde{f}_i(m+pk \mod d)v(k), \qquad \text{for } m = 1, 2, \ldots, d.$$

The reduction modulo d is intentionally emphasized. These operators satisfy conditions similar to those of aperiodic filters:

Proposition.

(1) $\tilde{F}_i\tilde{F}_j^* = 0$, *if* $i \neq j$,
(2) $\tilde{F}_i\tilde{F}_i^* = I_{d/p}$
(3) $\tilde{F}_i^*\tilde{F}_i$ *is a rank* d/p *orthogonal projection on* $\mathbf{R}^d$, *and for* $i \neq j$ *the ranges of* $\tilde{F}_i^*\tilde{F}_i$ *and* $\tilde{F}_j^*\tilde{F}_j$ *are orthogonal,*
(4) $\tilde{F}_0^*\tilde{F}_0 + \cdots + \tilde{F}_{p-1}^*\tilde{F}_{p-1} = I_d$

where I_d *is the identity on* $\mathbf{R}^d$.

Proof. The proof is nearly identical with the one in the aperiodic case. □

The decomposition suggested by equation (4) may be recursively applied to the p subspaces $\mathbf{R}^{d/p}$ by using additional filter families. For $d = p_1 \ldots p_L$ and $0 \le n < d$, we have a unique representation $n = n_1 + n_2p_1 + n_3p_2p_1 + \cdots + n_Lp_{L-1} \ldots p_1$, where $0 \le n_i < p_i$. This defines a one-to-one correspondence between $\{0, \ldots, d-1\}$ and an index set of L-tuples $I = \{(n_1, \ldots, n_L) : 0 \le n_i < p_i\}$. We can construct a basis of $\mathbf{R}^d$ whose elements are indexed by I. For $n = (n_1, \ldots, n_L) \in I$, define $\tilde{\mathbf{F}}_n^L = \tilde{F}_{n_L}^L \ldots \tilde{F}_{n_1}^1$, where $\tilde{F}^i$ is a family of p_i periodic filters. Then $\tilde{\mathbf{F}}_n^{L*}\tilde{\mathbf{F}}_n^L$ is an orthogonal projection onto a 1-dimensional subspace of $\mathbf{R}^d$. This is shown by induction on the rank in (3). Now let W_n^L be the range of this projection. The collection $\{u_n = \tilde{\mathbf{F}}_n^{L*}1 : n \in I\}$ of standard basis vectors of W_n^L will be an orthonormal basis of $\mathbf{R}^d$, and the map $\tilde{\mathbf{F}}_n^L : \mathbf{R}^d \to \mathbf{R}$ gives the component in the u_n direction.

As before, we are not limited to the basis defined by the index set I. Products of fewer than L filters form orthogonal projections onto a tree of subspaces of $\mathbf{R}^d$. A node arising from a product of m filters will correspond to the subspace $W_n^m = \tilde{\mathbf{F}}_n^{m*}\tilde{\mathbf{F}}_n^m\mathbf{R}^d$, where $n = n_1 + \cdots + n_mp_{m-1} \cdots p_1$ indexes a composition of m filters. The tree will

be nonhomogeneous in general, although all nodes i levels from the root will have the same number p_i of daughters. Define a *nonhomogeneous graph* as a finite union of nodes whose associated subtrees form a disjoint cover of some level $m \leq L$. A graph theorem holds for this tree of subspaces as well. It and its corollary may be stated as follows:

Theorem. *Every nonhomogeneous graph corresponds to an orthogonal decomposition of* $\mathbf{R}^d$. □

Corollary. *Graphs are in one-to-one correspondence with finite disjoint covers of* $[0,1)$ *by intervals of the form* $I_n^m = (p_1 \dots p_m)^{-1}[n, n+1)$. □

Any permutation of the prime factors of d gives a (possibly different) basis.

Smooth filters. Some filter sequences have a smoothness property:

Definition. *A summable sequence f is a smooth filter (of rank p) if there is a nonzero solution ϕ in $L^1(\mathbf{R}) \cap L^2(\mathbf{R}) \cap C^\infty(\mathbf{R})$ to the functional equation*

$$\phi(x) = p^{1/2} \sum_m f(m)\phi(px+m).$$

A filter will be said to have smoothness degree r if it satisfies this definition with C^∞ replaced by C^r. Daubechies has shown in [D] that finitely supported filters of any degree of smoothness may be constructed in the case $p = 2$. An obvious consequence is that smooth filters exist in the case $p = 2^q$. For arbitrary p, we may construct a filter family as above subject to additional constraints.

A continuous L^2 solution to the functional equation (3) always exists for a sequence f satisfying the three conditions at the top of this article. Its Fourier transform may be constructed by iteration:

$$\hat{\phi}(\xi) = \hat{\phi}(0) \prod_{i=k}^{\infty} m(\xi/p^k).$$

where $m(\xi) = p^{-1/2} \sum_k f(k) e^{-ik\xi}$ is the multiplier corresponding to the filter convolution in the integral equation. If ϕ is nonzero, then $\phi(0) \neq 0$. so it may be assumed that $\phi(0) = 1$. Now the sequence $\{f(k)|k|^\epsilon\}$ converges absolutely, so $m(\xi)$ is Hölder continuous of degree ϵ. But also, $m(0) = p^{1/2} \sum_k f(k) = 1$. so that for ξ near 0 the estimate $|m(\xi) - 1| < C|\xi|^\epsilon$ holds. This implies that the infinite product converges.

REFERENCES

[CW] R. R. Coifman and Y. Meyer, *Nouvelles bases othonormées de $L^2(\mathbf{R})$ ayant la structure du système de Walsh*, preprint, Yale University, New Haven (1989).

[D] Ingrid Daubechies, *Orthonormal bases of compactly supported wavelets*, Communications on Pure and Applied Mathematics **XLI** (1988), 909–996.

[Ma] Stephane G. Mallat, *A Theory for Multiresolution Signal Decomposition: The Wavelet Decomposition*, IEEE Transactions on Pattern Analysis and Machine Intelligence **11** (1989), 674–693.

III-1

Appendix III

Nonstandard Matrix Multiplication

Wave packets. Define wave packets over l^2 in the usual way. For a pair $P = \{p_i\}, Q = \{q_i\}$ of quadrature mirror filters (QMFs) satisfying the orthogonality and decay conditions stated in [CW], there is a unique solution to the functional equation

$$\phi(t) = \sqrt{2}\sum_{j\in\mathbf{Z}} p_j\phi(2t-j).$$

Put $w = w_{0,0,0} = \phi$, and define recursively

$$w_{2n,0,0}(t) = \sqrt{2}\sum_{j\in\mathbf{Z}} p_j w_{n,0,0}(2t-j),$$
$$w_{2n+1,0,0}(t) = \sqrt{2}\sum_{j\in\mathbf{Z}} q_j w_{n,0,0}(2t-j).$$

Then set $w_{nmk}(t) = 2^{m/2}w_{n00}(2^m t - k)$. Write $\mathcal{W}(\mathbf{R}) = \{w_{nmk} : n, m, k \in \mathbf{Z}\}$ for the collection of functions so defined, which we shall call *wave packets.*

The quadrature mirror filters P, Q may be chosen so that $\mathcal{W}(\mathbf{R})$ is dense in many common function spaces. With the minimal hypotheses of [CW], $\mathcal{W}(\mathbf{R})$ will be dense in $L^2(\mathbf{R})$. Using the Haar filters $P = \{1/\sqrt{2}, 1/\sqrt{2}\}, Q = \{1/\sqrt{2}, -1/\sqrt{2}\}$ produces $\mathcal{W}(\mathbf{R})$ which is dense in $L^p(\mathbf{R})$ for $1 < p < \infty$. Longer filters can generate smoother wave packets, so we can also produce dense subsets of Sobolev spaces, etc.

Basis subsets. Define a *basis subset* σ of the set of indices $\{(n, m, k) \in \mathbf{Z}^3\}$ to be any subcollection with the property that $\{w_{nmk} : (n, m, k) \in \sigma\}$ is a Hilbert basis for $L^2(\mathbf{R})$. We characterize basis subsets in [W1]. Abusing notation, we shall also refer to the collection of wave packets $\{w_{nmk} : (n, m, k) \in \sigma\}$ as a basis subset.

Since $L^2 \cap L^p$ is dense in L^p for $1 \le p < \infty$, with certain QMFs a basis subset will also be a basis for L^p. Likewise, for nice enough QMFs, it will be a Hilbert basis for the various Sobolev spaces.

Since $L^2(\mathbf{R}) \otimes L^2(\mathbf{R})$ is dense in $L^2(\mathbf{R}^2)$, the collection of vectors $\{w_X \otimes w_Y : w_X \in \mathcal{W}(X), w_Y \in \mathcal{W}(Y)\}$ is dense in the space of Hilbert-Schmidt operators. Call $\sigma \subset \mathbf{Z}^6$ a

basis subset if the collection $\{w_{n_X m_X k_X} \otimes w_{n_Y m_Y k_Y} : (n_X, m_X, k_X, n_Y, m_Y, k_Y) \in \sigma\}$ forms a Hilbert basis. Such two-dimensional basis subsets are characterized in [W2].

Ordering wave packets. Wave packets w_{nmk} can be totally ordered. We say that $w < w'$ if $(m, n, k) < (m', n', k')$. The triplets are compared lexicographically, counting the scale parameter m as most significant.

Tensor products of wave packets inherit this total order. Write $w_X = w_{n_X m_X k_X}$, etc. Then we will say that $w_X \otimes w_Y < w'_X \otimes w'_Y$ if $w_X < w'_X$, or else if $w_X = w'_X$ but $w_Y < w'_Y$. This is equivalent to

$$(m_X, n_X, k_X, m_Y, n_Y, k_Y) < (m'_X, n'_X, k'_X, m'_Y, n'_Y, k'_Y),$$

comparing lexicographically from left to right.

Define the *adjoint order* $<^*$ by exchanging X and Y indices, namely $w_X \otimes w_Y <^* w'_X \otimes w'_Y$ if and only if $w_Y \otimes w_X <^* w'_Y \otimes w'_X$. This is also a total order.

Projections. Let $\mathcal{W}^1$ denote the space of bounded sequences indexed by the three wave packet indices n, m, k. With the ordering above, we obtain a natural isomorphism between l^∞ and $\mathcal{W}^1$. There is also a natural injection $J^1 : L^2(\mathbf{R}) \hookrightarrow \mathcal{W}^1$ given by $c_{nmk} = \langle v, w_{nmk} \rangle$, for $v \in L^2(\mathbf{R})$ and $w_{nmk} \in \mathcal{W}(\mathbf{R})$. If σ is a basis subset, then the composition J^1_σ of J^1 with projection onto the subsequences indexed by σ is also injective. J^1_σ is an isomorphism of $L^2(\mathbf{R})$ onto $l^2(\sigma)$, which is defined to be the square summable sequences of $\mathcal{W}^1$ whose indices belong to σ.

We also have a map $R^1 : \mathcal{W}^1 \to L^2(\mathbf{R})$ defined by

$$R^1 c(t) = \sum_{(n,m,k) \in \mathbf{Z}^3} c_{nmk} w_{nmk}(t).$$

This map is defined and bounded on the closed subspace of $\mathcal{W}^1$ isomorphic to l^2 under the natural isomorphism mentioned above. In particular, R^1 is defined and bounded on the range of J^1_σ for every basis subset σ. The related restriction $R^1_\sigma : \mathcal{W}^1 \to L^2(\mathbf{R})$ defined by $R^1_\sigma c(t) = \sum_{(n,m,k)\in\sigma} c_{nmk} w_{nmk}(t)$ is a left inverse for J^1 and J^1_σ. In addition, $J^1 R^1_\sigma$ is a projection of $\mathcal{W}^1$. Likewise, if $\sum_i \alpha_i = 1$ and $R^1_{\sigma_i}$ is one of the above maps for each i, then $J^1 \sum_i \alpha_i R^1_{\sigma_i}$ is also a projection of $\mathcal{W}^1$. It is an orthogonal projection on any finite subset of $\mathcal{W}^1$.

Similarly, writing $\mathcal{W}^2$ for $\mathcal{W}^1 \times \mathcal{W}^1$, the ordering of tensor products gives a natural isomorphism between l^∞ and $\mathcal{W}^2$. The space $L^2(\mathbf{R}^2)$, i.e., the Hilbert-Schmidt operators, inject into this sequence space $\mathcal{W}^2$ in the obvious way, namely $M \mapsto$

$\langle M, w_{n_X m_X k_X} \otimes w_{n_Y m_Y k_Y} \rangle$. Call this injection J^2. If σ is a basis subset of $\mathcal{W}^2$, then the composition J^2_σ of J^2 with projection onto subsequences indexed by σ is also injective. J^2_σ is an isomorphism of $L^2(\mathbf{R}^2)$ onto $l^2(\sigma)$, the square summable sequences of $\mathcal{W}^2$ whose indices belong to σ.

The map $R^2 : \mathcal{W}^2 \to L^2(\mathbf{R}^2)$ given by $R^2 c(x,y) = \sum c_{XY} w_X(x) w_Y(y)$, is bounded on that subset of $\mathcal{W}^2$ naturally isomorphic to l^2. In particular, it is bounded on the range of J^2_σ for every basis subset σ.

We may also define the restrictions R^2_σ of R^2 to subsequences indexed by σ, defined by $R^2_\sigma c(x,y) = \sum_{(w_X, w_Y) \in \sigma} c_{XY} w_X(x) w_Y(y)$. There is one for each basis subset σ of $\mathcal{W}^2$. Then R^2_σ is a left inverse of J^2 and J^2_σ, and $J^2 R^2_\sigma$ is a projection of $\mathcal{W}^2$. As before, if $\sum_i \alpha_i = 1$ and σ_i is a basis subset for each i, then $J^2 \sum_i R^2_{\sigma_i}$ is also a projection of $\mathcal{W}^2$. It is an orthogonal projection on any finite subset of $\mathcal{W}^2$.

Applying operators to vectors. For definiteness, let X and Y be two named copies of $\mathbf{R}$. Let $v \in L^2(X)$ be a vector, whose coordinates with respect to wave packets form the sequence $J^1 v = \{\langle v, w_X \rangle : w_X \in \mathcal{W}(X)\}$.

Let $M : L^2(X) \to L^2(Y)$ be a Hilbert-Schmidt operator. Its matrix coefficients with respect to the complete set of tensor products of wave packets form the sequence $J^2 M = \{\langle M, w_X \otimes w_Y \rangle : w_X \in \mathcal{W}(X), w_Y \in \mathcal{W}(Y)\}$. We obtain the identity

$$\langle Mv, w_Y \rangle = \sum_{w_X \in \mathcal{W}(X)} \langle M, w_X \otimes w_Y \rangle \langle v, w_X \rangle$$

This identity generalizes to a linear action of $\mathcal{W}^2$ on $\mathcal{W}^1$ defined by

$$c(v)_{nmk} = \sum_{(n'm'k')} c_{nmkn'm'k'} v_{n'm'k'}.$$

Now, images of operators form a proper submanifold of $\mathcal{W}^2$. Likewise, images of vectors form a submanifold $\mathcal{W}^1$. We can lift the action of M on v to these larger spaces via the commutative diagram

$$\begin{array}{ccc} \mathcal{W}^1 & \xrightarrow{J^2_\sigma M} & \mathcal{W}^1 \\ {\scriptstyle J^1}\uparrow & & \downarrow{\scriptstyle R^1} \\ L^2(\mathbf{R}) & \xrightarrow[M]{} & L^2(\mathbf{R}) \end{array}$$

The significance of this lift is that by a suitable choice of σ we can reduce the complexity of the map $J^2_\sigma M$, and therefore the complexity of the operator application.

Composing operators. Let X, Y, Z be three named copies of $\mathbf{R}$. Suppose that $M : L^2(X) \to L^2(Y)$ and $N : L^2(Y) \to L^2(Z)$ are Hilbert-Schmidt operators. We have the identity

$$\langle NM, w_X \otimes w_Z \rangle = \sum_{w_Y \in \mathcal{W}(Y)} \langle N, w_Y \otimes w_Z \rangle \langle M, w_X \otimes w_Y \rangle.$$

This generalizes to an action of $\mathcal{W}^2$ on $\mathcal{W}^2$, which is defined by the formula

$$c(d)_{nmkn'm'k'} = \sum_{n''m''k''} d_{nmkn''m''k''} c_{n''m''k''n'm'k'},$$

where c and d are sequences in $\mathcal{W}^2$. Using J^2, we can lift multiplication by N to an action on these larger spaces via the commutative diagram

$$\begin{tikzcd}
\mathcal{W}^2 \arrow[r, "J^2_\sigma N"] & \mathcal{W}^2 \arrow[d, "R^2"] \\
L^2(\mathbf{R}^2) \arrow[u, "J^2"] \arrow[r, "N"'] & L^2(\mathbf{R}^2)
\end{tikzcd}$$

Again, by a suitable choice of σ the complexity of the operation may be reduced to below that of ordinary operator composition.

Operation counts: transforming a vector. Suppose that M is a non-sparse operator of rank r. Ordinary multiplication of a vector by M takes at least $O(r^2)$ operations, with the minimum achievable only by representing M as a matrix with respect to the bases of its r-dimensional domain and range.

On the other hand, the injection J^2 will require $O(r^2[\log r]^2)$ operations, and each of J^1 and R^1 require $O(r \log r)$ operations. For a fixed basis subset σ of $\mathcal{W}^2$, the application of $J^2_\sigma M$ to $J^1 v$ requires at most $\#|J^2_\sigma M|$ operations, where $\#|U|$ denotes the number of nonzero coefficients in U. We may choose our wavelet library so that $\#|J^2_\sigma M| = O(r^2)$. Thus the multiplication method described above costs an initial investment of $O(r^2[\log r]^2)$, plus at most an additional $O(r^2)$ per right-hand side. Thus the method has asymptotic complexity $O(r^2)$ per vector in its exact form, as expected for what is essentially multiplication by a conjugated matrix.

We can obtain lower complexity if we take into account the finite accuracy of our calculation. Given a fixed matrix of coefficients C, write C_δ for the same matrix with all coefficients set to 0 whose absolute values are less than δ. By the continuity of the Hilbert-Schmidt norm, for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|C - C_\delta\|_{HS} < \epsilon$. Given

M and ϵ as well as a library of wave packets, we can choose a basis subset $\sigma \subset \mathcal{W}^2$ so as to minimize $\#|(J_\sigma^2 M)_\delta|$. The choice algorithm has complexity $O(r^2[\log r]^2)$, as shown in [W2]. For a certain class of operators, there is a library of wave packets such that for every fixed $\delta > 0$ we have

$$\text{(S)} \qquad \#|(J_\sigma^2 M)_\delta| = O(r \log r),$$

with the constant depending, of course, on δ. We will characterize this class, give examples of members, and find useful sufficient conditions for membership in it. For the moment, call this class with property S the *sparsifiable* Hilbert-Schmidt operators $\mathcal{S}$. By the estimate above, finite-precision multiplication by sparsifiable rank-r operators has asymptotic complexity $O(r \log r)$.

Operation counts: composing two operators. Suppose that M and N are rank-r operators. Standard multiplication of N and M has complexity $O(r^3)$. The complexity of injecting N and M into $\mathcal{W}^2$ is $O(r^2[\log r]^2)$. The action of $J_\sigma^2 N$ on $J^2 M$ has complexity $O(\sum_{nmk} \#|J_\sigma^2 N_{YZ} : (n_Y, m_Y, k_Y) = (n, m, k)| \#|J^2 M_{XY} : (n_Y, m_Y, k_Y) = (n, m, k)|)$. The second factor is a constant $r \log r$, while the first when summed over all nmk is exactly $\#|J_\sigma^2 N|$. Thus the complexity of the nonstandard multiplication algorithm, including the conjugation into the basis set σ, is $O(\#|J_\sigma^2 N| \, r \log r)$. Since the first factor is r^2 in general, the complexity of the exact algorithm is $O(r^3 \log r)$ for generic matrices, reflecting the extra cost of conjugating into the basis set σ.

For the approximate algorithm, the complexity is $O(\#|(J_\sigma^2 N)_\delta| \, r \log r)$. For the sparsifiable matrices, this can be reduced by a suitable choice of σ to a complexity of $O(r^2[\log r]^2)$ for the complete algorithm. Since choosing σ and evaluating J_σ^2 each have this complexity, it is not possible to do any better by this method.

Reference

[CW] –, *Appendix II.*
[W2] –, *Appendix V.*

Appendix IV

Entropy of a Vector Relative to a Decomposition

Let $v \in H$ $\|v\| = 1$ and assume

$$H = \oplus \sum H_i$$

an orthogonal direct sum. We define

$$\varepsilon^2(v, \{H_i\}) = -\sum \|v_i\|^2 \ell u \|v_i\|^2$$

as a measure of distance between v and the orthogonal decomposition.

ε^2 is characterized by the Shannon equation which is a version of Pythagoras' theorem.

Let

$$\begin{aligned} H &= \oplus(\sum H^i) \oplus (\sum H_j) \\ &= H_+ \oplus H_- \end{aligned}$$

H^i and H_j give orthogonal decomposition $H_+ = \sum H^i$ $H_- = \sum H_j$. Then

$$\varepsilon^2(v; \{h^i, h_j\}) = \varepsilon^2(v, \{H_+, H_-\} + \|v_+\|^2 \varepsilon^2 \left(\frac{v_+}{\|v++\|}, \{H^i\}\right) + \|v_-\|^2 \varepsilon^2 \left(\frac{v_-}{\|v\|}, \{H_j\}\right)$$

This is Shannon's equation for entropy (if we interpret as in quantum mechanics $\|P_{H_+} v\|^2$ as the "probability" of v to be in the subspace H_+).

This equation enables us to search for a smallest entropy space decomposition of a given vector. We need the following $H = H_1 \oplus H_2$.

$$H_1 = \oplus \sum H^i = \oplus \sum K^j.$$

H_1 has two decompositions in H^i or K^j.

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Lemma 1. *Let $v \in H$ with $\|v\|$ and*

$$v_1 = P_{H_1} v \qquad v_1' = \frac{v_1}{\|v_1\|}$$

Assume also that $\varepsilon^2(v_1', \{H^i\}) < \varepsilon^2(v_1', \{K^i\})$ then, if $H_2 = \oplus \sum L^j$ we have

$$\varepsilon^2(v, \{H^i, L^j\}) < \varepsilon^2(v, \{K^j, L^i\})$$

Proof. By Shannon's equation

$$\begin{aligned}
\varepsilon^2(v, \{H^i, L^j\}) &= \varepsilon^2(v, \{H_1, H_2\}) + \|v_1\|^2 \varepsilon^2(v_1', \{H^i\}) \\
&\quad + \|v_2\|^2 \varepsilon^2(v_2', \{L^j\}) \\
&< \varepsilon^2(v, \{H_1, H_2\}) + \|v_1\|^2 \varepsilon^2(v_1', \{K^j\}) + \|v_2\|^2 \varepsilon^2(v_2', \{L^j\}) \\
&= \varepsilon^2(v, \{K^j, L^i\}).
\end{aligned}$$

Corollary. *Assume $\varepsilon^2(v_1', \{H^i\})$ is the smallest entropy obtained for some collection of decompositins of H_1 and similarly, $\varepsilon^2(v_2', \{L^j\})$ is minimal. Then $\varepsilon^2(v; \{H^i, L^j\})$ is minimal for the direct sum of these collections.*

We consider the following generic example on $L^2(\mathbf{R}_+)$.

Let I denote a dyadic interval of the form $(2^j k, 2^j(k+1)]$, $j \geq 0$ $k \geq 0$, and $\{I_\alpha\}$ a disjoint cover over $(0, \infty)$ consisting of dyadic intervals. We let $H_{I_\alpha} = L^2(I_\alpha)$ on which we chose an orthonormal basis $\{e^{I_\alpha}_{\alpha,k}\}$ α fixed (say trig polynomials $\exp(2\pi i \frac{x}{2^j}) \chi_{I_\alpha}(x)$.) and consider $\{e^{I_\alpha}_{\alpha,k}\}$ as an orthonormal basis of $L^2(\mathbf{R}^+)$. Thus

$$L^2(\mathbf{R}^+ = \sum H_{I_\alpha} = \sum_\alpha \sum_k \{e_{\alpha,k}\}$$

Given a vector v we wish to find I_α such that $\varepsilon^2(v, \{e_{\alpha,k}\})$ is minimal. In order to find I_α we use a stopping time argument. Starting with intervals of length one $I_\ell = (\ell, \ell+1]$. We pick a dyadic interval of length two which contains halves $I_1 I_2$ of length one, i.e. $J = I_1 \cup I_2$. We compart

$$\varepsilon^2(v\chi_J, \{e^J_k\}) \quad \text{with} \quad \varepsilon^2(v\chi_J, \{e^{I_1}_k\}\{e^{I_2}_k\})$$

and pick the basis given the smallest entropy leading to a cover of $L^2(\mathbf{R})$ by intervals of length one and two. We now consider dyadic intervals K of length 4 and compare

$$\varepsilon^2(v\chi_K \{e^K_j\}) \quad \text{with} \quad \varepsilon^2(v\chi_K; \{e^{I_\alpha}_K\})$$

where I_α form a cover of K by dyadic intervals of length one or two selected previously to minimize ε^2 on each half of K.

(If the vector function v has bounded support we restrict our attention only to dyadic intervals contained in the smallest dyadic interval containing the support of v) and continue this procedure up to this largest scale. We claim that the final partition I_α and corresponding basis provides the minimal entropy decomposition. In fact, this is an immediate consequence of Lemma 1 which shows that given the optimal minimum entropy partition any refinement corresponds to worse entropy.

Appendix V

Higher-Dimensional Best Basis Selection

Introduction. We introduce a method of coding by orthogonal functions which may be used to compress digitized pictures or sequences of pictures, matrices and linear operators, and general sampled functions of several variables. The method selects a most efficient orthogonal representation from among a large number of possibilities. The efficiency functional need only be additive across direct sum decompositions. We present a description of the method for pictures, namely functions of two variables, using Shannon entropy as the efficiency functional, and mean-square deviation as the error criterion.

Best basis method. In Appendix II is developed a method for generating a library of orthogonal vectors in $\mathbf{R}^n$ (for large n) together with a notion of admissible subsets of these vectors. Admissible subsets form orthonormal bases of wavelet-packets, which because of their homogeneous tree structure may be rapidly searched for functional extrema. We can use a family of orthonormal vectors well suited to representing functions of 2 variables. These are products of quadrature mirror filters, as defined below:

Let $\{p_k\}, \{q_k\}$ belong to l^1, and define two decimating convolution operators $P : l^2 \to l^2$, $Q : l^2 \to l^2$ as follows:

$$Pf_k = \sum_{j=-\infty}^{\infty} p_j f_{j+2k}, \qquad Qf_k = \sum_{j=-\infty}^{\infty} q_j f_{j+2k}.$$

P and Q are called *quadrature mirror filters* if they satisfy an orthogonality condition:

$$PQ^* = QP^* = 0,$$

where P^* denotes the adjoint of P, and Q^* the adjoint of Q. They are further called *perfect reconstruction filters* if they satisfy the condition

$$P^*P + Q^*Q = I,$$

where I is the identity operator. These conditions translate to restrictions on the sequences $\{p_k\}, \{q_k\}$. Let m_0, m_1 be (bounded) functions defined by

$$m_0(\xi) = \sum_{k=-\infty}^{\infty} p_k e^{ik\xi}, \qquad m_1(\xi) = \sum_{k=-\infty}^{\infty} q_k e^{ik\xi}.$$

Then P, Q are quadrature mirror filters if and only if the matrix below is unitary for all ξ:

$$\begin{pmatrix} m_0(\xi) & m_0(\xi+\pi) \\ m_1(\xi) & m_1(\xi+\pi) \end{pmatrix}$$

This fact is proved in [D].

Now we can define a number of orthogonal 2-dimensional convolution-decimation filters in terms of P and Q. Four of them are simply tensor products of the pair of quadrature mirror filters, as in the construction of 2-dimensional wavelets of Meyer [M].

$$F_0 \stackrel{\text{def}}{=} P \otimes P, \qquad F_0 v(x,y) = \sum_{i,j} v(i,j) p_{2x+i} p_{2y+j}$$

$$F_1 \stackrel{\text{def}}{=} P \otimes Q, \qquad F_1 v(x,y) = \sum_{i,j} v(i,j) p_{2x+i} q_{2y+j}$$

$$F_2 \stackrel{\text{def}}{=} Q \otimes P, \qquad F_2 v(x,y) = \sum_{i,j} v(i,j) q_{2x+i} p_{2y+j}$$

$$F_3 \stackrel{\text{def}}{=} Q \otimes Q, \qquad F_3 v(x,y) = \sum_{i,j} v(i,j) p_{2x+i} p 2y + j$$

There are also pairs of extensions of one dimensional filters:

$$P_Y \stackrel{\text{def}}{=} I \otimes P, \qquad P_Y v(x,y) = \sum_{i,j} v(i,j) \delta_{x,i} p_{2y+j} = \sum_j v(x,j) p_{2y+j}.$$

$$Q_Y \stackrel{\text{def}}{=} I \otimes Q, \qquad Q_Y v(x,y) = \sum_{i,j} v(i,j) \delta_{x,i} q_{2y+j} = \sum_j v(x,j) q_{2y+j}$$

$$P_X \stackrel{\text{def}}{=} P \otimes I, \qquad P_X v(x,y) = \sum_{i,j} v(i,j) p_{2x+i} \delta_{y,j} = \sum_i v(i,y) p_{2x+i}$$

$$Q_X \stackrel{\text{def}}{=} Q \otimes I, \qquad Q_X v(x,y) = \sum_{i,j} v(i,j) q_{2x+i} \delta_{y,j} = \sum_i v(i,y) q_{2x+i}$$

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These convolution-decimations have the following adjoints:

$$F_0^*v(x,y) = \sum_{i,j} v(i,j)p_{2i+x}p_{2j+y}$$
$$F_1^*v(x,y) = \sum_{i,j} v(i,j)p_{2i+x}q_{2j+y}$$
$$F_2^*v(x,y) = \sum_{i,j} v(i,j)q_{2i+x}p_{2j+y}$$
$$F_3^*v(x,y) = \sum_{i,j} v(i,j)p_{2i+x}p_{2j+y}$$

$$P_Y^*v(x,y) = \sum_{i,j} v(i,j)\delta_{x,i}p_{2j+y} = \sum_j v(x,j)p_{2j+y}$$
$$Q_Y^*v(x,y) = \sum_{i,j} v(i,j)\delta_{x,i}q_{2j+y} = \sum_j v(x,j)q_{2j+y}$$
$$P_X^*v(x,y) = \sum_{i,j} v(i,j)p_{2i+x}\delta_{y,j} = \sum_i v(i,y)p_{2i+x}$$
$$Q_X^*v(x,y) = \sum_{i,j} v(i,j)q_{2i+x}\delta_{y,j} = \sum_i v(i,y)q_{2i+x}$$

The orthogonality relations for this collection are as follows:

$$F_nF_m^* = \delta_{nm}I$$
$$I = F_0^*F_0 \oplus F_1^*F_1 \oplus F_2^*F_2 \oplus F_3^*F_3$$
$$P_XP_X^* = P_YP_Y^* = Q_XQ_X^* = Q_YQ_Y^* = I$$
$$P_XQ_X^* = Q_XP_X^* = P_YQ_Y^* = Q_YP_Y^* = 0$$
$$I = P_X^*P_X \oplus Q_X^*Q_X = P_Y^*P_Y \oplus Q_Y^*Q_Y$$

By a "picture" we will mean any function $S = S(x,y) \in l^2(\mathbf{Z}^2)$. The space $l^2(\mathbf{Z}^2)$ of pictures may be decomposed into a partially ordered set $\mathbf{W}$ of subspaces $W(n_X, n_Y, m_X. m_Y)$, where $m_X \geq 0$, $m_Y \geq 0$, $0 \leq n_X < 2^{m_X}$, and $0 \leq n_Y < 2^{m_Y}$. These are the images of orthogonal projections composed of products of convolution-

decimations. Put $W(0,0,0,0) = l^2$, and define recursively

$$W(2n_X + i, 2n_Y + j, m_X + 1, m_Y + 1) = F^*_{2i+j} F_{2i+j} W(n_X, n_Y, m_X, m_Y),$$

$$W(2n_X, n_Y, m_X + 1, m_Y) = P^*_X P_X W(n_X, n_Y, m_X, m_Y),$$

$$W(2n_X + 1, n_Y, m_X + 1, m_Y) = Q^*_X Q_X W(n_X, n_Y, m_X, m_Y),$$

$$W(n_X, 2n_Y, m_X, m_Y + 1) = P^*_Y P_Y W(n_X, n_Y, m_X, m_Y),$$

$$W(n_X, 2n_Y + 1, m_X, m_Y + 1) = Q^*_X Q_X W(n_X, n_Y, m_X, m_Y).$$

These subspaces may be partially ordered by a relation which we define recursively as well. We say W is a *precursor* of W' (write $W \preceq W'$) if they are equal or if $W' = G^*GW$ for a convolution-decimation G in the set $\{F_0, F_1, F_2, F_3, P_X, P_Y, Q_X, Q_Y\}$. We also say that $W \preceq W'$ if there is a finite sequence $V_1, \ldots, V_n$ of subspaces in $\mathbf{W}$ such that $W \preceq V_1 \preceq \cdots \preceq V_n \preceq W'$. This is well defined, since each application of G^*G increases at least one of the indices m_X or m_Y.

While $\{\mathbf{W}, \preceq\}$ is not a tree, it may be made into a tree if we select a subset of the relation $\preceq$. We will say that $W = W(n_X, n_Y, m_X, m_Y)$ is a *principal precursor* of $W' = W(n'_X, n'_Y, m'_X, m'_Y)$ (and write $W \prec W'$) if one of the following holds:

(1) $m_Y = m_X$, and $W' = G^*GW$ for $G \in \{F_0, F_1, F_2, F_3, P_X, P_Y, Q_X, Q_Y\}$, or
(2) $m_Y < m_X$, and $W' = G^*GW$ for $G \in \{P_X, Q_X\}$, or
(3) $m_Y > m_X$, and $W' = G^*GW$ for $G \in \{P_Y, Q_Y\}$.

Further, we will say that $W \prec W'$ if there is a finite sequence $V_0, \ldots, V_n$ of subspaces in $\mathbf{W}$ with $W \prec V_0 \prec \cdots \prec V_n \prec W'$. The relation $\prec$ is well defined, since it is a subrelation of $\preceq$, and it is not hard to see that every subspace $W \in \mathbf{W}$ has a unique first principal precursor. Therefore, $\{\mathbf{W}, \prec\}$ forms a (nonhomogeneous) tree, with $W(0,0,0,0)$ at its root.

Subspaces of a single principal precursor $W \in \mathbf{W}$ will be called its *children*. By the orthogonality condition,

$$\begin{aligned}
&\text{(F)} & W &= F^*_0 F_0 W \oplus F^*_1 F_1 W \oplus F^*_2 F_2 W \oplus F^*_3 F_3 W \\
&\text{(X)} & &= P^*_X P_X W \oplus Q^*_X Q_X W \\
&\text{(Y)} & &= P^*_Y P_Y W \oplus Q^*_Y Q_Y W.
\end{aligned}$$

The right hand side contains all the children of W, divided into the groups "F," "X." and "Y." Each labelled group of children provides a one-step orthogonal decomposition of W, and in general we will have three subsets of the children to choose from.

The coordinates with respect to the standard basis of $W(n_X, n_Y, m_X, m_Y)$ form the sequence $G_1 \ldots G_m W(0,0,0,0)$, where $m = \max\{m_X, m_Y\}$, and the particular filters $G_1 \ldots G_m$ are determined uniquely by n_X and n_Y. This is described in Appendix II, attached hereto. Therefore we can express in standard coordinates the orthogonal projections of $W(0,0,0,0)$ onto the complete tree of subspaces **W** by recursively convolving and decimating with the filters.

Relation with one-dimensional wave packets. Let w_{nmk} be a one-dimensional wave packet at sequency n, scale m and position k, in the notation of Appensix II. Then the element in the (k_X, k_Y) position of the subspace $W(n_X, n_Y, m_X, m_Y)$, at the index (x, y), may be written as $w_{n_X, m_X, k_X}(x) w_{n_Y, m_Y, k_Y}(y)$, which is evidently the tensor product of two one-dimensional wave packets. This is easily seen from the construction of $W(n_X, n_Y, m_X, m_Y)$: in the x-direction, there will be a total of m_X convolution-decimations in the order determined by n_X, with the result translated to position k_X, and similarly in the y-direction.

We will use the notation $w \otimes v$ for the tensor product of two one-dimensional wave packets, with the understanding that the second factor depends on the y-coordinate. Since the one-dimensional wave packets are themselves a redundant spanning set, their tensor products contain a redundancy of bases for $l^2\mathbf{R}^2$. We can search this collection of bases efficiently for a best-adapted basis, using any additive measure of information, in a manner only slightly more complicated than for the one dimensional case.

Selecting a best basis. Let $S = S(x, y)$ be a picture, and let **W** be a tree of wavelet packets. Choose an additive measure of information as described in Appendix II, and attribute to each node $W(n_X, n_Y, m_X, m_Y)$ the measure of information contained in the coordinates of S with respect to the wavelet packets it contains. For example, we may use Shannon entropy,

$$\mathcal{H}(W) = \sum_{k_X, k_Y} p^2 \log p^2,$$

where $p = \langle S, w_{n_X m_X k_X} \otimes w_{n_Y m_Y k_Y} \rangle$, and $W = W(n_X, m_X, n_Y, m_Y)$. We will choose an arbitrary maximum level in the tree **W**, and mark all of its nodes as "kept." Proceeding up from this level to the root, we will compare $\mathcal{H}(W)$ for a node W of the tree **W** to the minimum of $\sum_{W \prec W' \in F} \mathcal{H}(W')$, $\sum_{W \prec W' \in X} \mathcal{H}(W')$, and $\sum_{W \prec W' \in Y} \mathcal{H}(W')$. If $\mathcal{H}(W)$ is less, then mark W as "kept" and mark as "not kept" all nodes W' with $W \prec W'$; otherwise, mark$W$ as "not kept," but attribute to it the minimum of the entropies of its children. When this procedure terminates at the root, the nodes marked "kept" will comprise an orthogonal collection of wavelet packets.

It is not necessary to mark all descendants of a "kept" parent as not kept. The complexity of the search algorithm is $O(n \log n)$ if we never change the status of descendants, but instead take for the orthogonal collection only those nodes marked "kept" which have no ancestors marked "kept." These may be listed efficiently by indexing the tree in the preorder or depth-first order.

Error estimates for the best basis. Let $\mathcal{H}(S)$ denote the entropy of the picture S in the best basis found above. This quantity will be found attributed to node $W(0,0,0,0)$ at the end of the search. It is related to the classical Shannon entropy $\mathcal{H}_0$ by the equation

$$\mathcal{H}_0(S) = \|S\|^{-2}\mathcal{H}(S) + \log \|S\|^2$$

The largest $\exp \mathcal{H}_0(S) = \|S\|^2 \exp \mathcal{H}(S)/\|S\|^2$ terms of the wavelet packet expansion for S contain essentially all the energy of the original picture. Mean square error bounds for specific classes of signals are provided in Appendix IV.

References

[D] Ingrid Daubechies, *Orthonormal bases of compactly supported wavelets*, Communications on Pure and Applied Mathematics **XLI** (1988), 909–996.

[M] Yves Meyer, *De la recherche petroliére à la geometrie des espaces de Banach en passant par les paraproduits*, Seminaire equations aux derivées partielles 1985-1986, École Polytechnique. Palaiseaux.

CLAIMS:

1. A method for encoding and decoding an input signal, comprising the steps of:

applying combinations of dilations and translations of a wavelet to the input signal to obtain processed values;

computing the information costs of the processed values;

selecting, as encoded signals, an orthogonal group of processed values, the selection being dependent on the computed information costs; and

decoding the encoded signals to obtain an output signal.

2. The method as defined by claim 1, wherein said wavelet has a plurality of vanishing moments.

3. The method as defined by claim 1 or 2, further comprising transmitting the encoded signals, and receiving the transmitted encoded signals before the decoding thereof.

4. The method as defined by claim 3, further comprising storing the encoded signals, and reading the stored encoded signals before the decoding thereof.

5. The method as defined by claim 2, wherein said step of applying combinations of dilations and translations of the wavelet to the input signal to obtain processed values comprises correlating said combinations of dilations and translations of the wavelet with the input signal.

6. The method as defined by claim 2 or 5, wherein combinations of dilations and translations of the wavelet are designated as wavelet-packets, and wherein the step of applying wavelet-packets to the input signal to obtain processed values includes: generating a tree of processed values, the tree having successive levels obtained by applying to the input signal, for a given level, wavelet-packets which are combinations of the wavelet-packets applied at a previous level.

7. The method as defined by claim 6, wherein the steps of computing information costs and selecting an orthogonal group of processed values includes performing said computing at a number of different levels of said tree, and performing said selecting

from among the different levels of the tree.

8. The method as defined by claim 2 or 7, wherein said step of selecting an orthogonal group of processed values comprises selecting an orthogonal group having a minimal information cost.

9. The method as defined by claim 8, wherein said step of selecting an orthogonal group of processed values includes generating encoded signals which represent said processed values in conjunction with their respective locations in said tree.

10. A method for encoding an input signal, comprising the steps of:

selecting a wavelet having a plurality of vanishing moments;

applying combinations of dilations and translations of the wavelet to the input signal to obtain processed values; and

selecting, as encoded signals, an orthogonal group of processed values.

11. The method as defined by claim 10, further comprising decoding the encoded signals.

12. The method as defined by claim 10 or 11, further comprising transmitting the encoded signals, and receiving the transmitted encoded signals before the decoding thereof.

13. The method as defined by claim 10, wherein said step of applying combinations of dilations and translations of the wavelet to the input signal to obtain processed values comprises correlating said combinations of dilations and translations of the wavelet with the input signal.

14. A method for encoding an input signal, comprising the steps of:

applying combinations of dilations and translations of a wavelet to the input signal to obtain processed values;

computing the information costs of the processed values; and

selecting, as encoded signals, an orthogonal group of processed values, the selection being dependent on the computed information costs.

15. The method as defined by claim 14, wherein said

wavelet has a plurality of vanishing moments.

16. The method as defined by claim 15, further comprising transmitting the encoded signals, and receiving the transmitted encoded signals before the decoding thereof.

17. The method as defined by claim 15, further comprising storing the encoded signals.

18. The method as defined by claim 15, wherein said step of applying combinations of dilations and translations of the wavelet to the input signal to obtain processed values comprises correlating said combinations of dilations and translations of the wavelet with the input signal.

19. The method as defined by claim 15 or 18, wherein combinations of dilations of the wavelet are designated as wavelet-packets, and wherein the step of applying wavelet-packets to the input signal to obtain processed values includes: generating a tree of processed values, the tree having successive levels obtained by applying to the input signal, for a given level, wavelet-packets which are combinations of the wavelet-packets applied at a previous level.

20. The method as defined by claim 19, wherein the steps of computing information costs and selecting an orthogonal group of processed values includes performing said computing at a number of different levels of said tree, and performing said selecting from among the different levels of the tree.

21. The method as defined by claim 15, wherein said step of selecting an orthogonal group of processed values comprises selecting an orthogonal group having a minimal information cost.

22. The method as defined by claim 20, wherein said step of selecting an orthogonal group of processed values comprises selecting an orthogonal group having a minimal information cost.

23. The method as defined by claim 22, wherein said step of selecting an orthogonal group of processed values includes generating encoded signals which represent said processed values in conjunction with their respective locations in said tree.

24. For use in a system which receives an encoded signal obtained by: applying combinations of dilations and translations of a wavelet having a plurality of vanishing moments to the input

signal to obtain processed values; and selecting, as encoded signals, an orthogonal group of processed values; a decoding method comprising: sequentially applying combinations of dilations and translations of a wavelet having a plurality of vanishing moments to the encoded signals to decode said encoded signals; and outputting the decoded result.

25. The decoding method as defined by claim 24, wherein the encoded signals include identification of the selected orthogonal group of processed values, and further comprising the step of determining said identification, and performing the sequential decoding procedure in accordance with said identification.

26. Apparatus for encoding an input signal, comprising:

means for applying combinations of dilations and translations of the wavelet to the input signal to obtain processed values;

means for computing the information costs of the processed values;

means for selecting, as encoded signals, an orthogonal group of processed values, the selection being dependent on the computed information costs.

27. Apparatus as defined by claim 26, wherein said wavelet has a plurality of vanishing moments.

28. Apparatus as defined by claim 27, wherein said means for applying combinations of dilations and translations of the wavelet to the input signal to obtain processed values comprises means for correlating said combinations of dilations and translations of the wavelet with the input signal.

29. Apparatus for encoding an input signal, comprising:

means for applying combinations of dilations and translations of a wavelet having a plurality of vanishing moments to the input signal to obtain processed values; and

means for selecting, as encoded signals, an orthogonal group of processed values.

30. For use in a system which receives an encoded signal obtained by: applying combinations of dilations and translations of a wavelet having a plurality of vanishing moments to the input signal to obtain processed values; and selecting, as encoded

signals, an orthogonal group of processed values; a decoding apparatus comprising: means for sequentially applying combinations of dilations and translations of a wavelet having a plurality of vanishing moments to the encoded signals to decode said encoded signals; and means for outputting the decoded result.

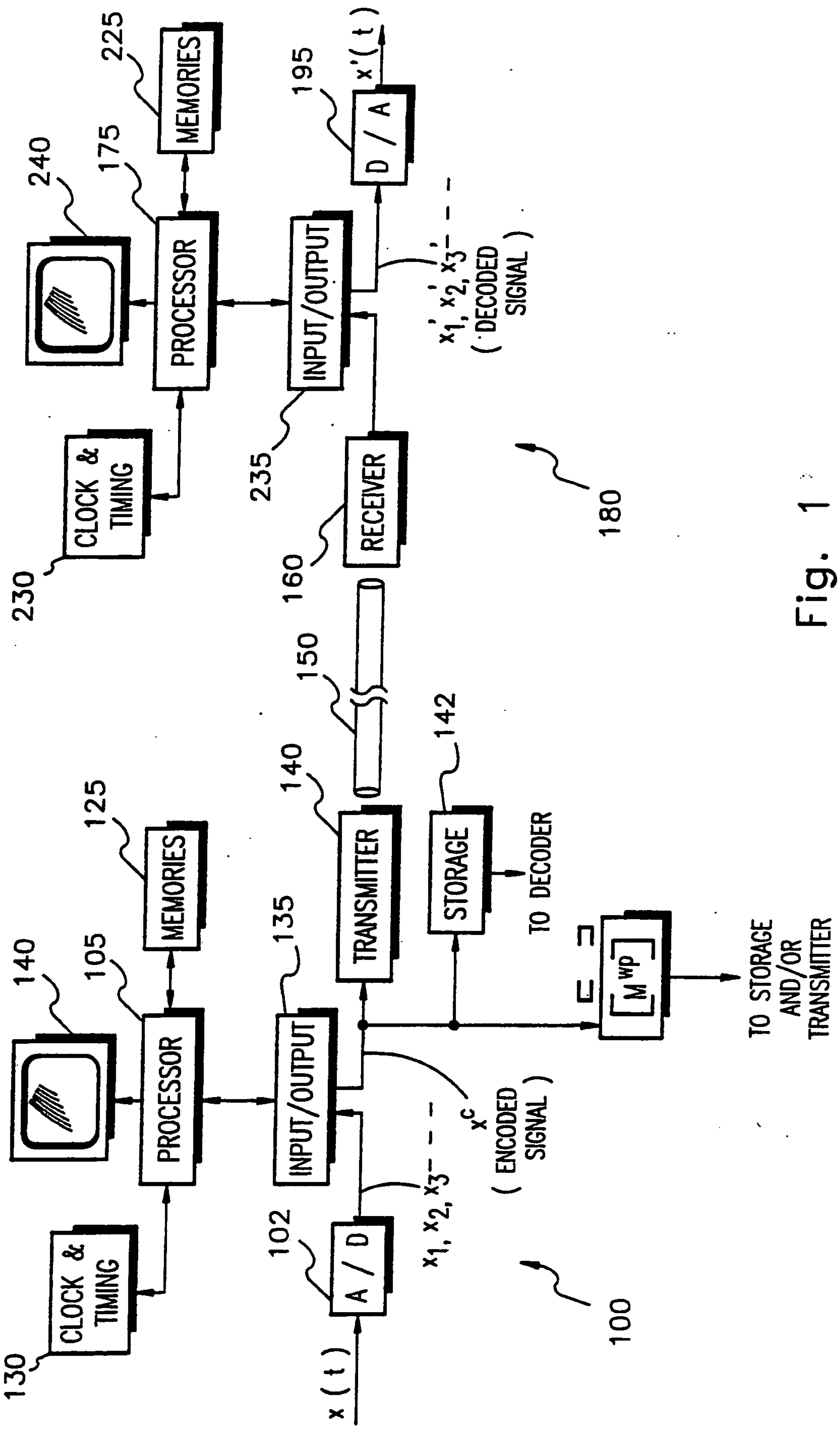

CLOCK & TIMING
130
PROCESSOR
105
140
MEMORIES
125
INPUT/OUTPUT
135
TRANSMITTER
140
STORAGE
142
TO DECODER
A / D
102
x (t)
x_1, x_2, x_3 - - -
x^c
(ENCODED SIGNAL)
$[M^{wp}]$
TO STORAGE AND/OR TRANSMITTER
100
150
RECEIVER
160
CLOCK & TIMING
230
PROCESSOR
175
240
MEMORIES
225
INPUT/OUTPUT
235
D / A
195
x'(t)
x_1', x_2', x_3' - - -
(DECODED SIGNAL)
180

Fig. 1

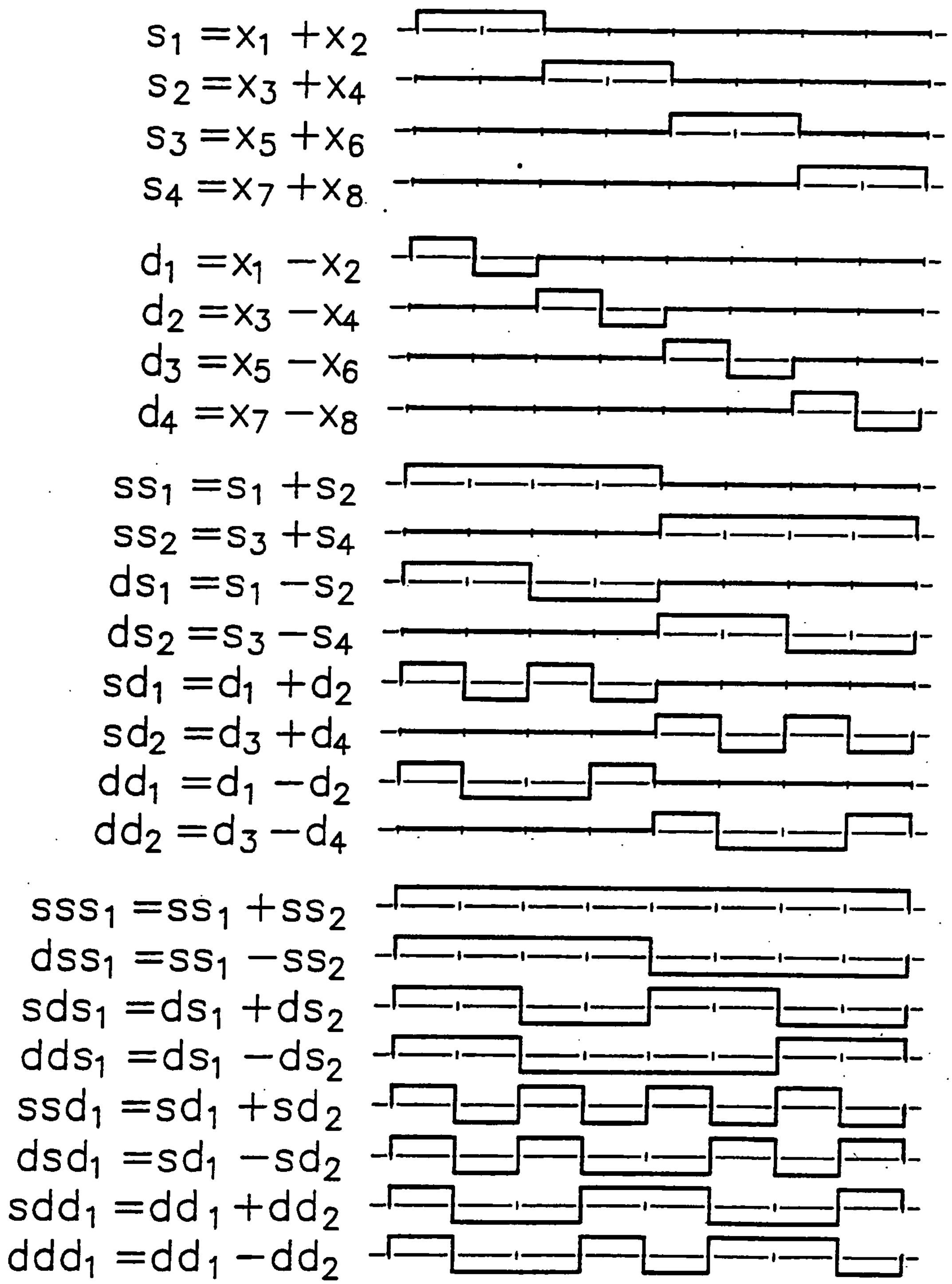

$s_1 = x_1 + x_2$
$s_2 = x_3 + x_4$
$s_3 = x_5 + x_6$
$s_4 = x_7 + x_8$
$d_1 = x_1 - x_2$
$d_2 = x_3 - x_4$
$d_3 = x_5 - x_6$
$d_4 = x_7 - x_8$
$ss_1 = s_1 + s_2$
$ss_2 = s_3 + s_4$
$ds_1 = s_1 - s_2$
$ds_2 = s_3 - s_4$
$sd_1 = d_1 + d_2$
$sd_2 = d_3 + d_4$
$dd_1 = d_1 - d_2$
$dd_2 = d_3 - d_4$
$sss_1 = ss_1 + ss_2$
$dss_1 = ss_1 - ss_2$
$sds_1 = ds_1 + ds_2$
$dds_1 = ds_1 - ds_2$
$ssd_1 = sd_1 + sd_2$
$dsd_1 = sd_1 - sd_2$
$sdd_1 = dd_1 + dd_2$
$ddd_1 = dd_1 - dd_2$

Fig. 2

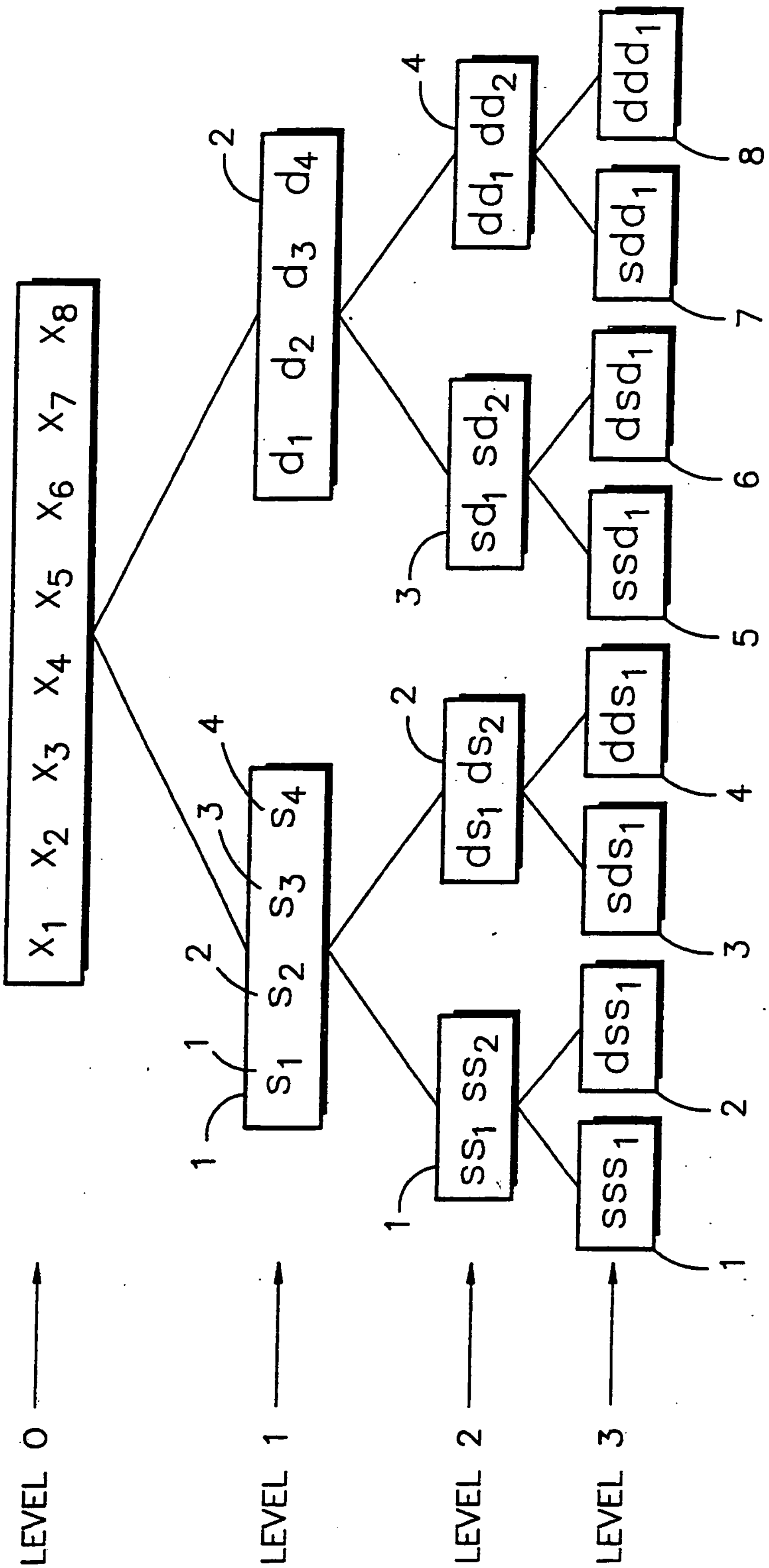

LEVEL 0
LEVEL 1
LEVEL 2
LEVEL 3
x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8
s_1 s_2 s_3 s_4
d_1 d_2 d_3 d_4
ss_1 ss_2
ds_1 ds_2
sd_1 sd_2
dd_1 dd_2
sss_1
dss_1
sds_1
dds_1
ssd_1
dsd_1
sdd_1
ddd_1
1
2
3
4
5
6
7
8

Fig. 3

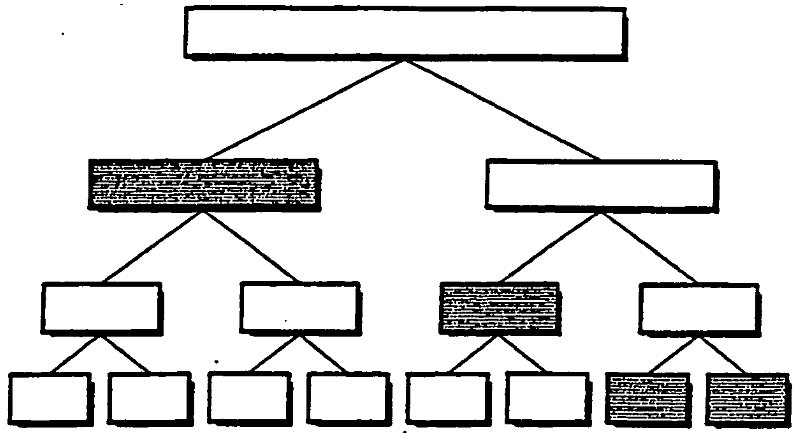


Fig. 4A

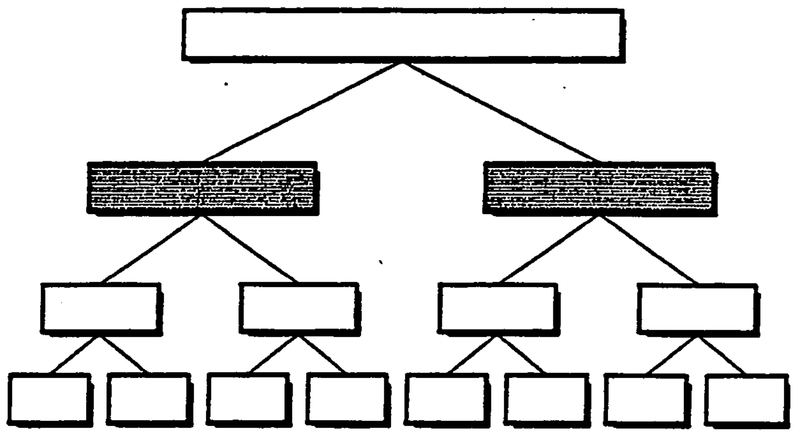


Fig. 4B

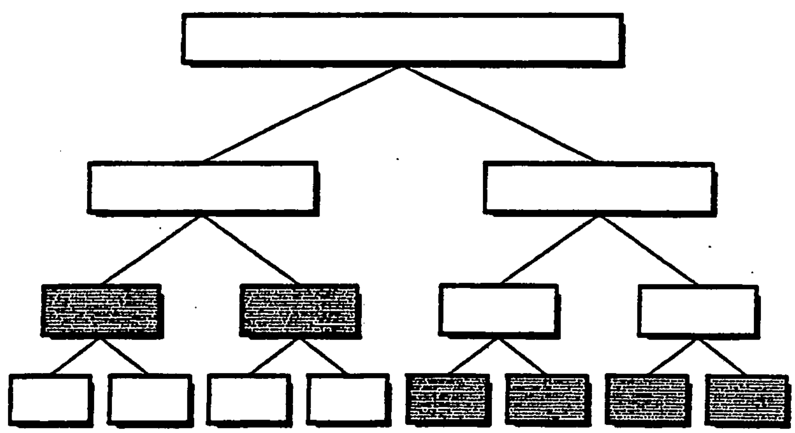


Fig. 4C

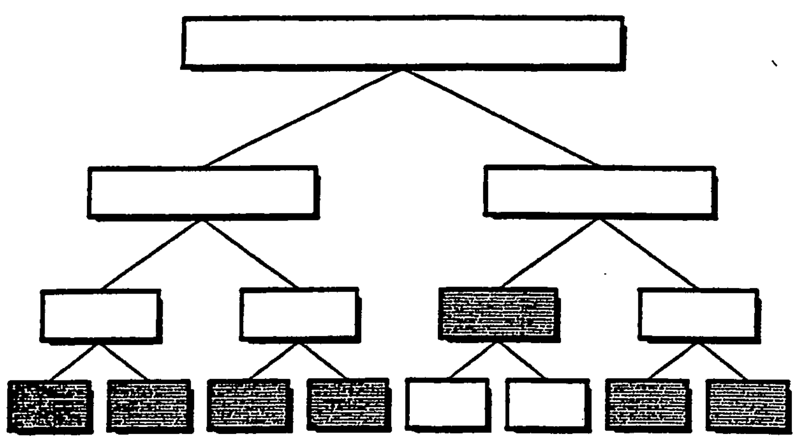


Fig. 4D

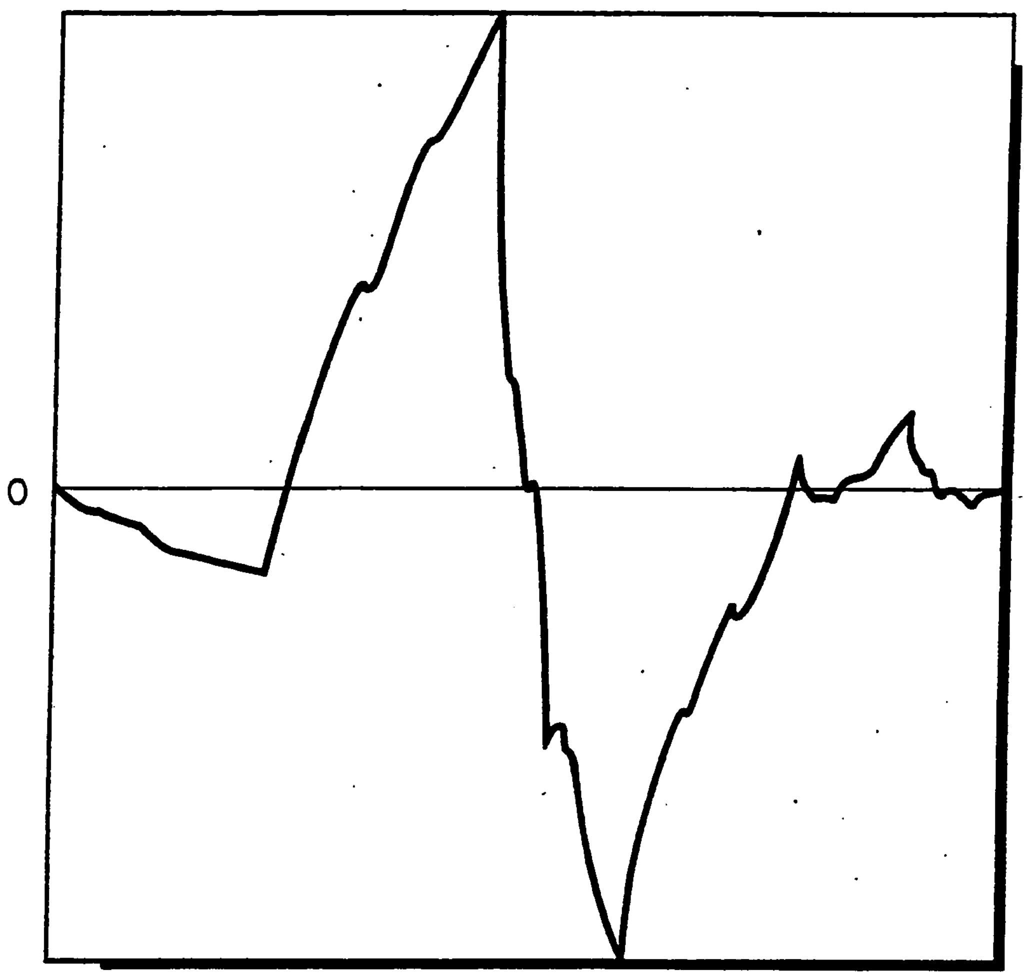


Fig. 5

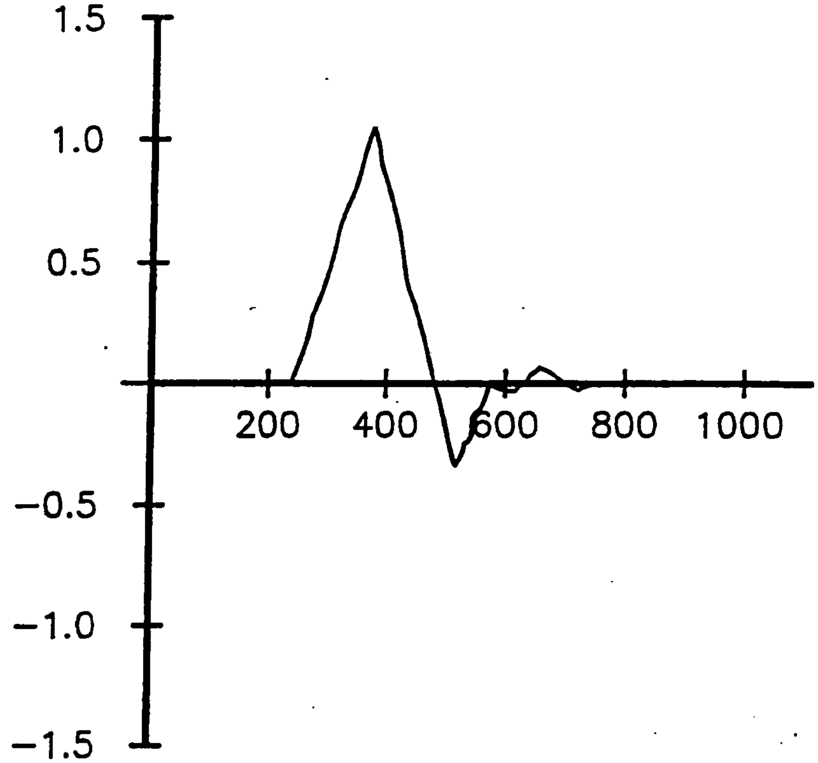

1.5
1.0
0.5
−0.5
−1.0
−1.5
200
400
600
800
1000

Fig. 6

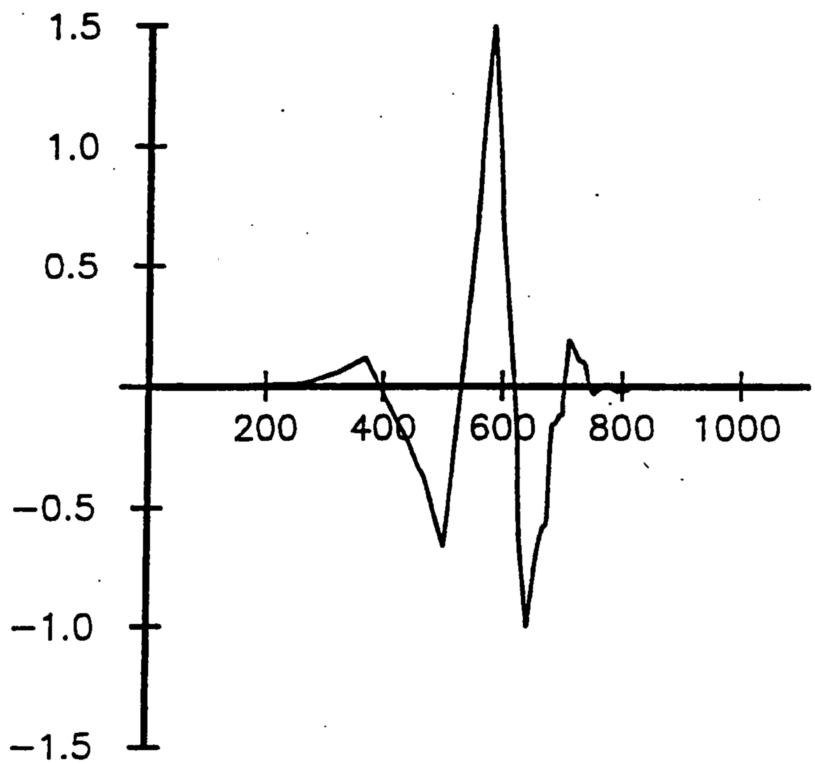

1.5
1.0
0.5
−0.5
−1.0
−1.5
200
400
600
800
1000

Fig. 7

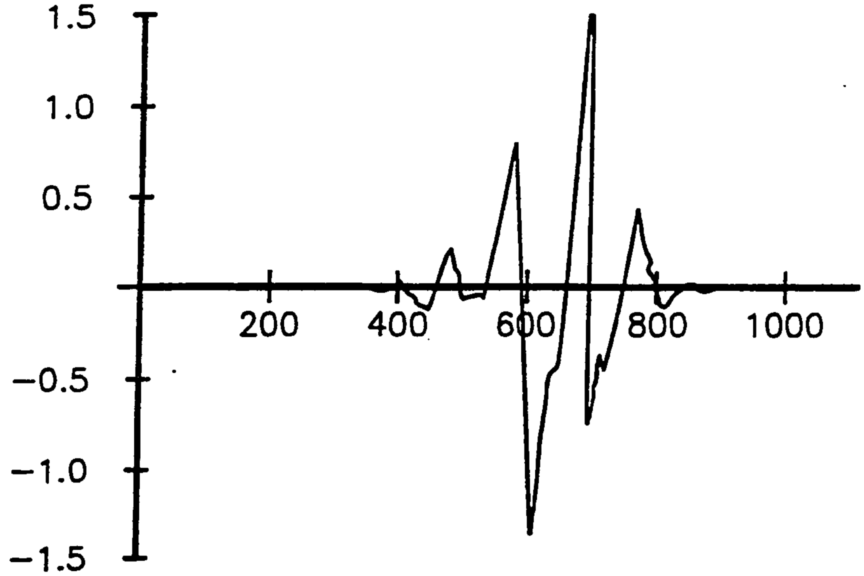


Fig. 8

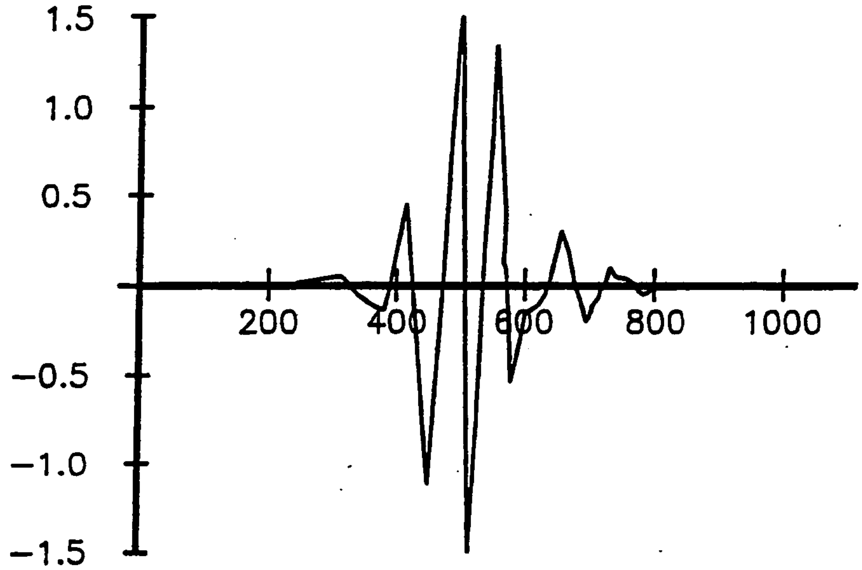


Fig. 9

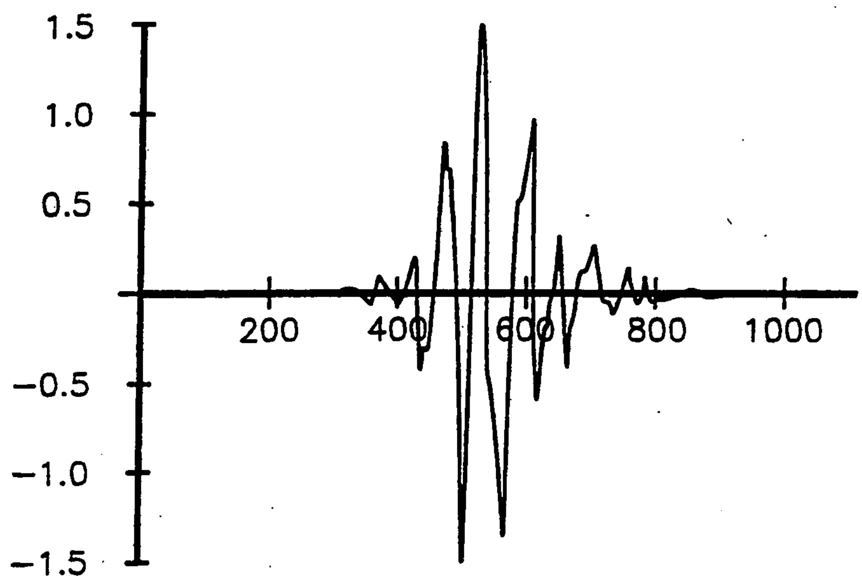


Fig. 10

Fig. 11

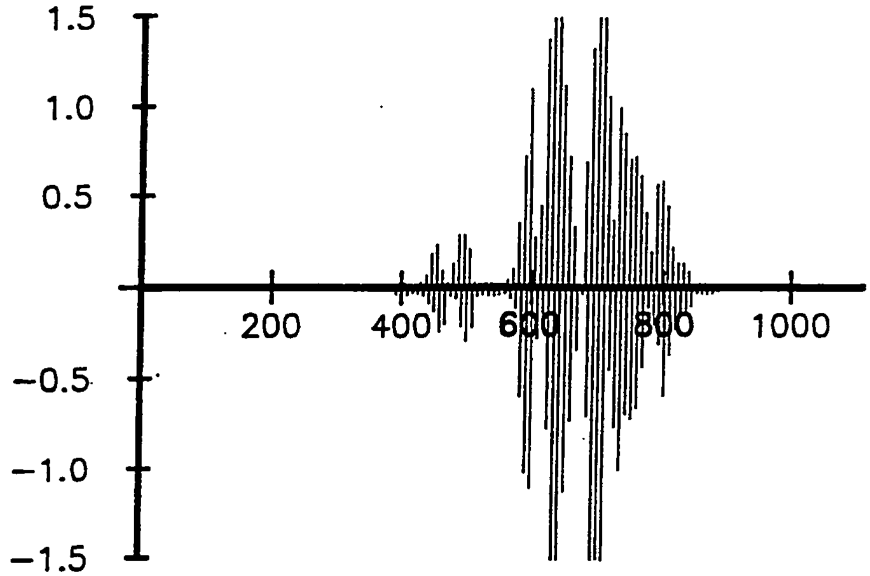


Fig. 12

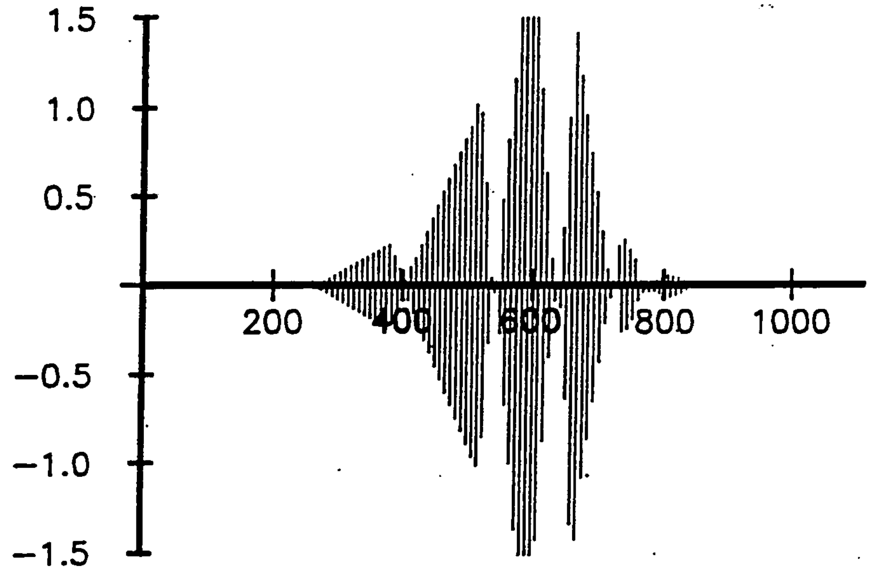
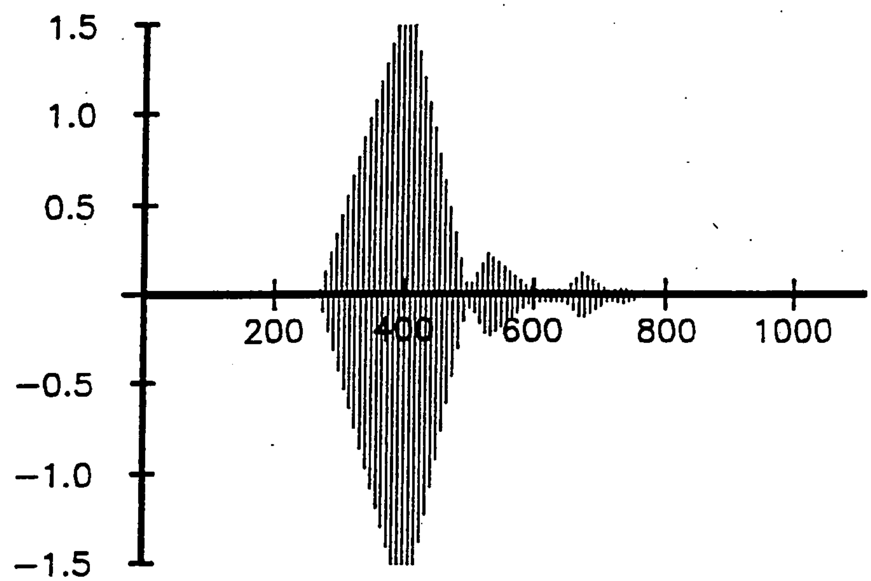


Fig. 13



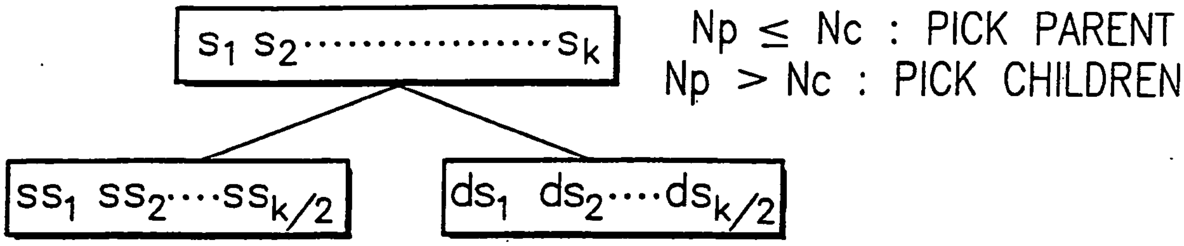


Fig. 14

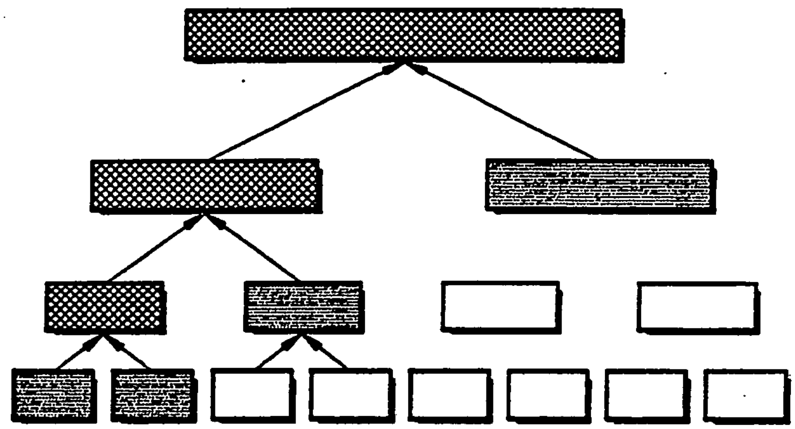


Fig. 15A

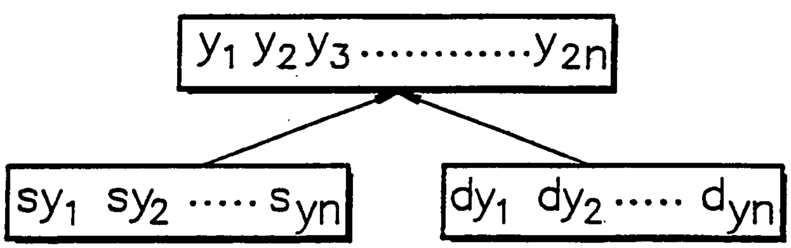


FiG. 15B

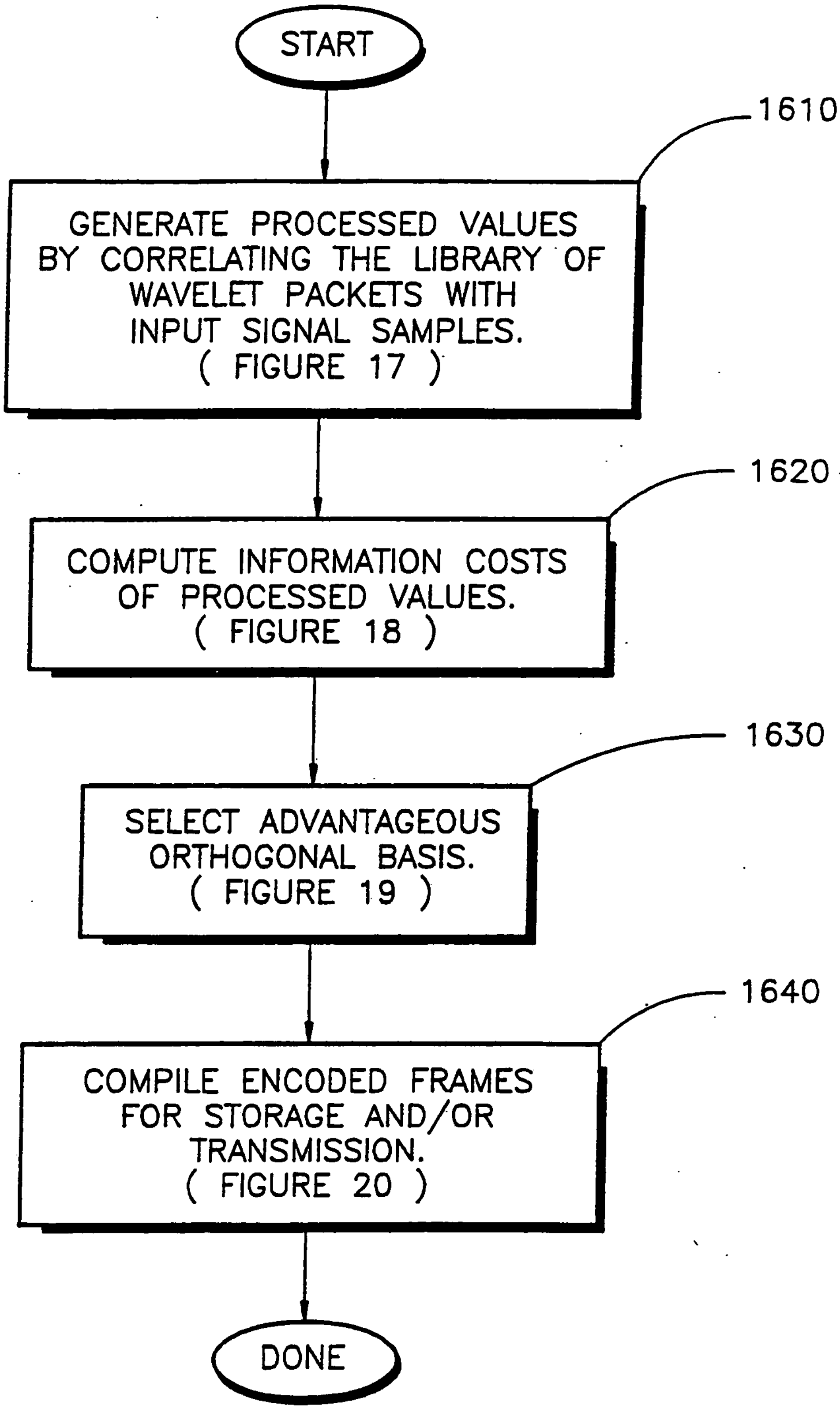

START
1610
GENERATE PROCESSED VALUES BY CORRELATING THE LIBRARY OF WAVELET PACKETS WITH INPUT SIGNAL SAMPLES. (FIGURE 17)
1620
COMPUTE INFORMATION COSTS OF PROCESSED VALUES. (FIGURE 18)
1630
SELECT ADVANTAGEOUS ORTHOGONAL BASIS. (FIGURE 19)
1640
COMPILE ENCODED FRAMES FOR STORAGE AND/OR TRANSMISSION. (FIGURE 20)
DONE

Fig. 16

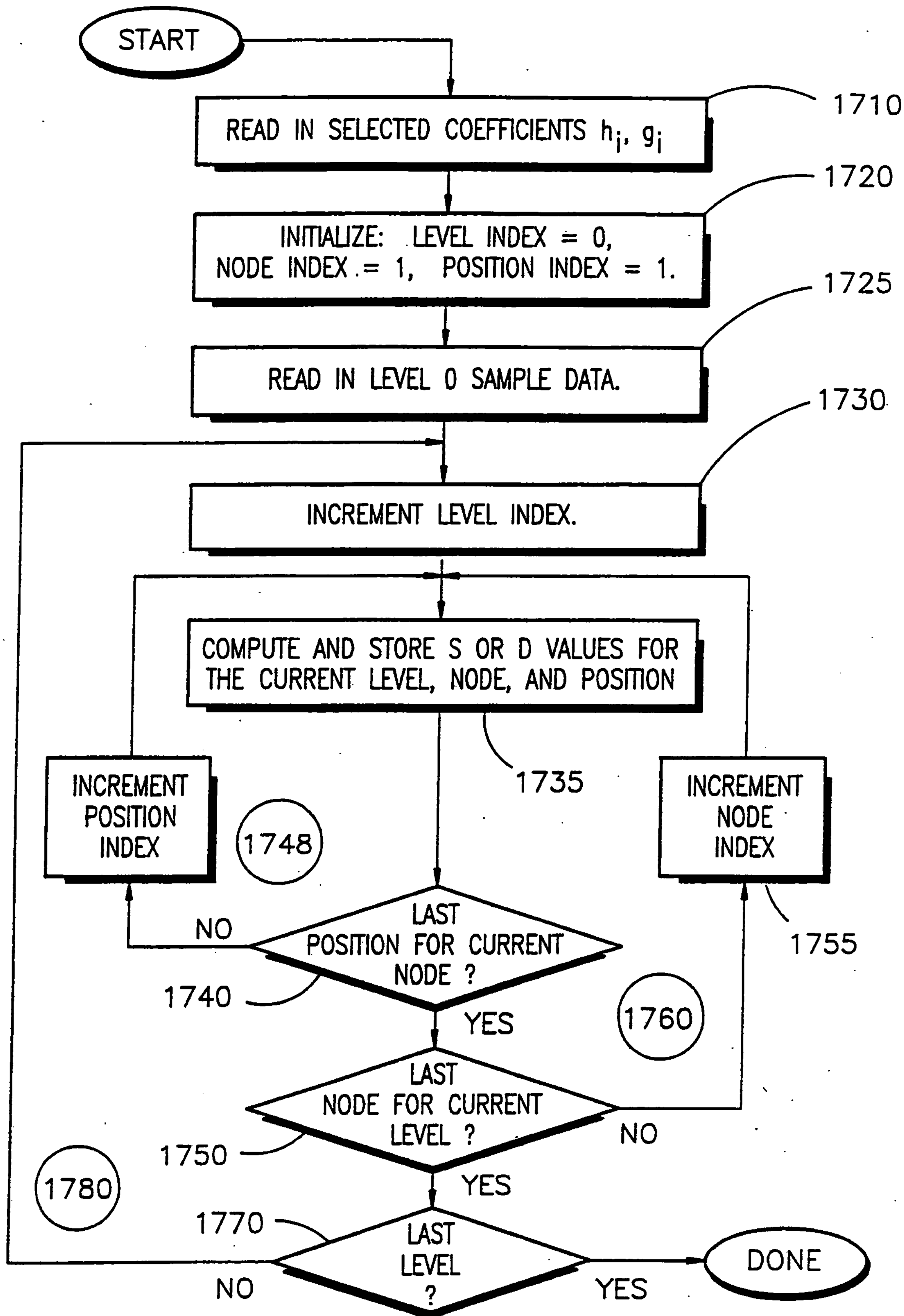

START
READ IN SELECTED COEFFICIENTS h_i, g_i
1710
INITIALIZE: LEVEL INDEX = 0,
NODE INDEX = 1, POSITION INDEX = 1.
1720
READ IN LEVEL 0 SAMPLE DATA.
1725
INCREMENT LEVEL INDEX.
1730
COMPUTE AND STORE S OR D VALUES FOR
THE CURRENT LEVEL, NODE, AND POSITION
1735
INCREMENT POSITION INDEX
1748
INCREMENT NODE INDEX
1755
LAST POSITION FOR CURRENT NODE ?
NO
YES
1740
1760
LAST NODE FOR CURRENT LEVEL ?
NO
YES
1750
1780
1770
LAST LEVEL ?
NO
YES
DONE

Fig. 17

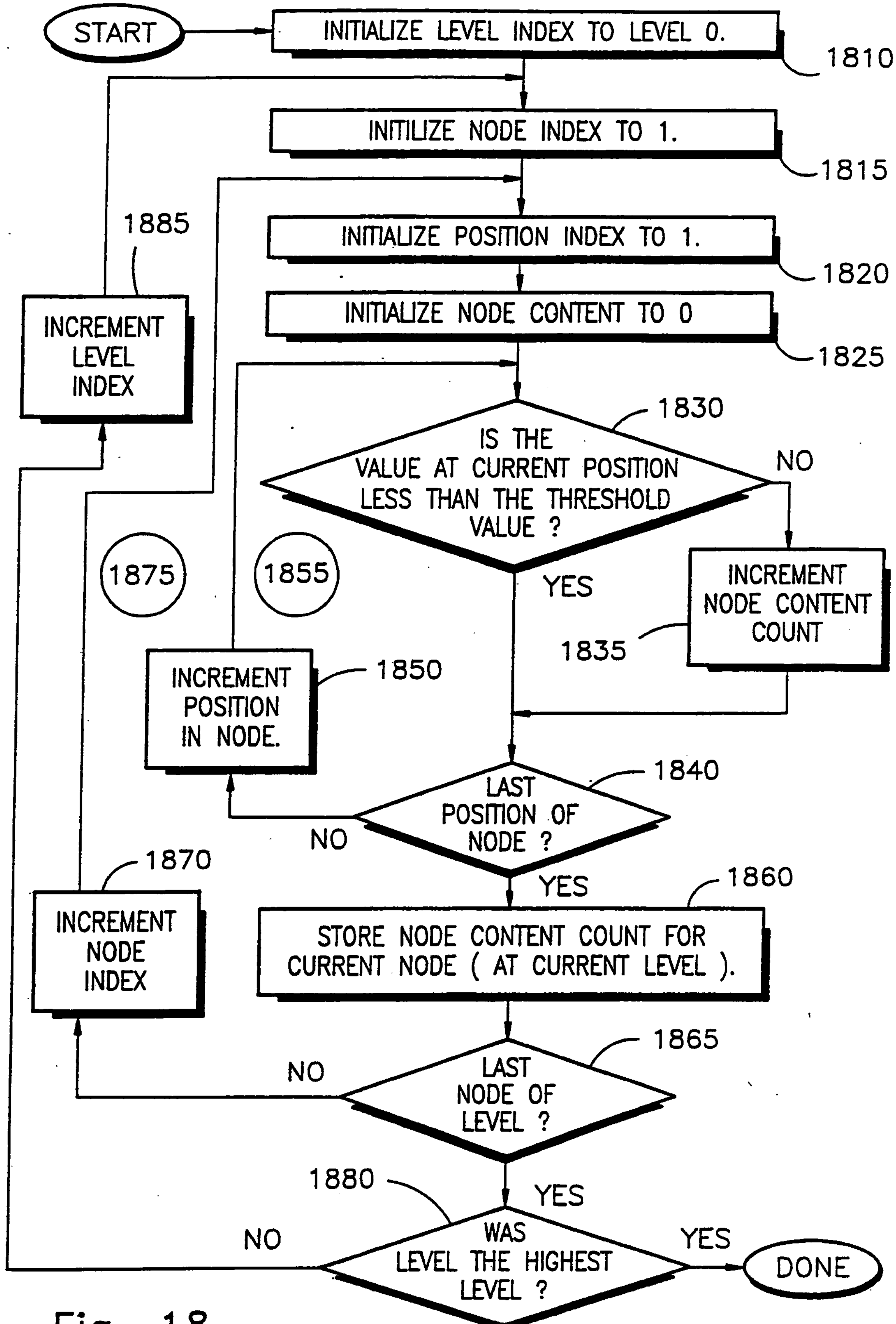

START
INITIALIZE LEVEL INDEX TO LEVEL 0.
1810
INITILIZE NODE INDEX TO 1.
1815
INITIALIZE POSITION INDEX TO 1.
1820
INITIALIZE NODE CONTENT TO 0
1825
1885
INCREMENT LEVEL INDEX
1830
IS THE VALUE AT CURRENT POSITION LESS THAN THE THRESHOLD VALUE ?
NO
YES
1875
1855
INCREMENT NODE CONTENT COUNT
1835
INCREMENT POSITION IN NODE.
1850
1840
LAST POSITION OF NODE ?
NO
YES
1870
1860
INCREMENT NODE INDEX
STORE NODE CONTENT COUNT FOR CURRENT NODE (AT CURRENT LEVEL).
1865
LAST NODE OF LEVEL ?
NO
YES
1880
WAS LEVEL THE HIGHEST LEVEL ?
NO
YES
DONE

Fig. 18

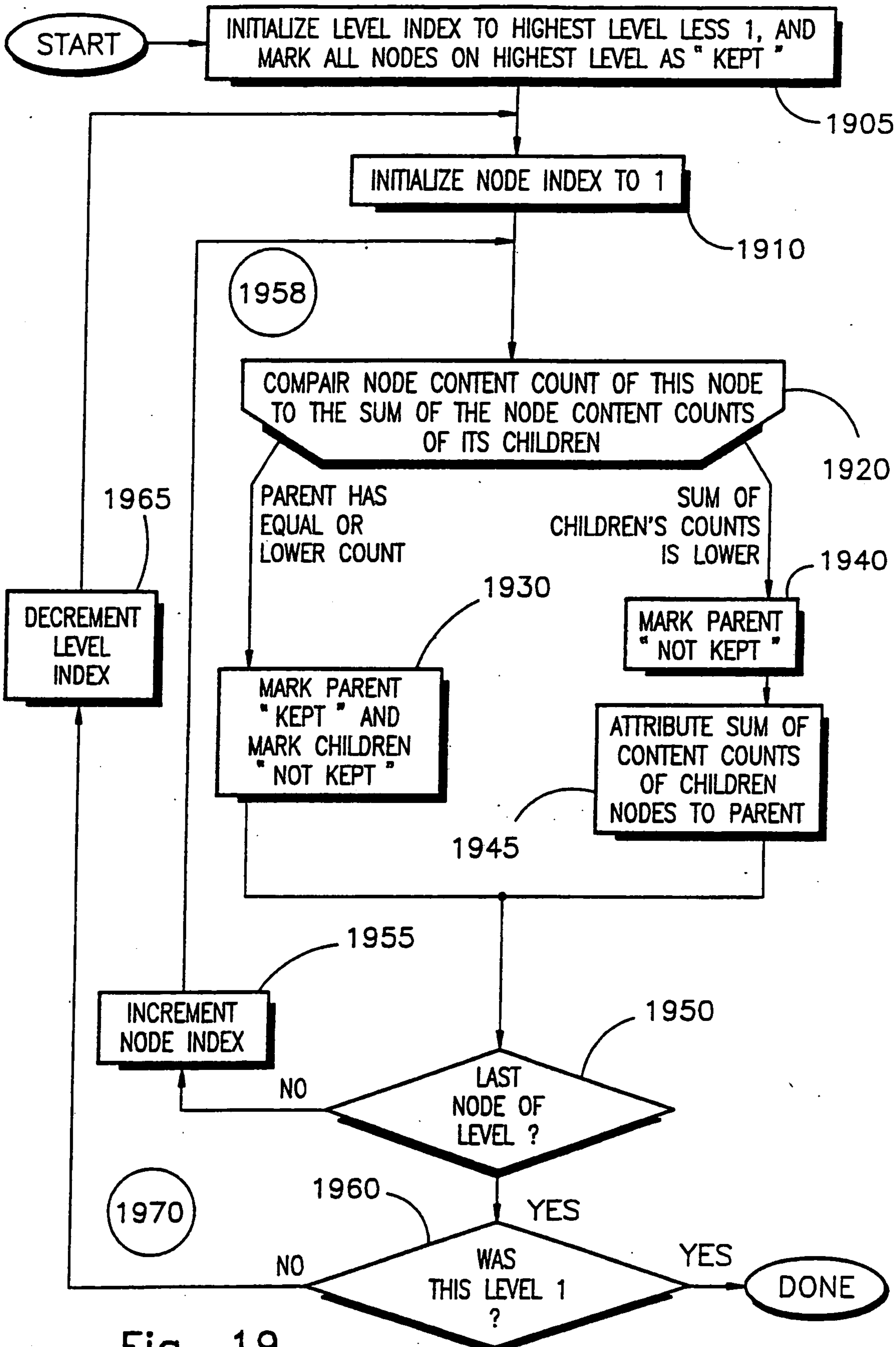

START
INITIALIZE LEVEL INDEX TO HIGHEST LEVEL LESS 1, AND MARK ALL NODES ON HIGHEST LEVEL AS " KEPT "
1905
INITIALIZE NODE INDEX TO 1
1910
1958
COMPAIR NODE CONTENT COUNT OF THIS NODE TO THE SUM OF THE NODE CONTENT COUNTS OF ITS CHILDREN
1920
PARENT HAS EQUAL OR LOWER COUNT
SUM OF CHILDREN'S COUNTS IS LOWER
1965
DECREMENT LEVEL INDEX
1930
MARK PARENT " KEPT " AND MARK CHILDREN " NOT KEPT "
1940
MARK PARENT " NOT KEPT "
ATTRIBUTE SUM OF CONTENT COUNTS OF CHILDREN NODES TO PARENT
1945
1955
INCREMENT NODE INDEX
1950
NO
LAST NODE OF LEVEL ?
YES
1960
1970
NO
WAS THIS LEVEL 1 ?
YES
DONE

Fig. 19

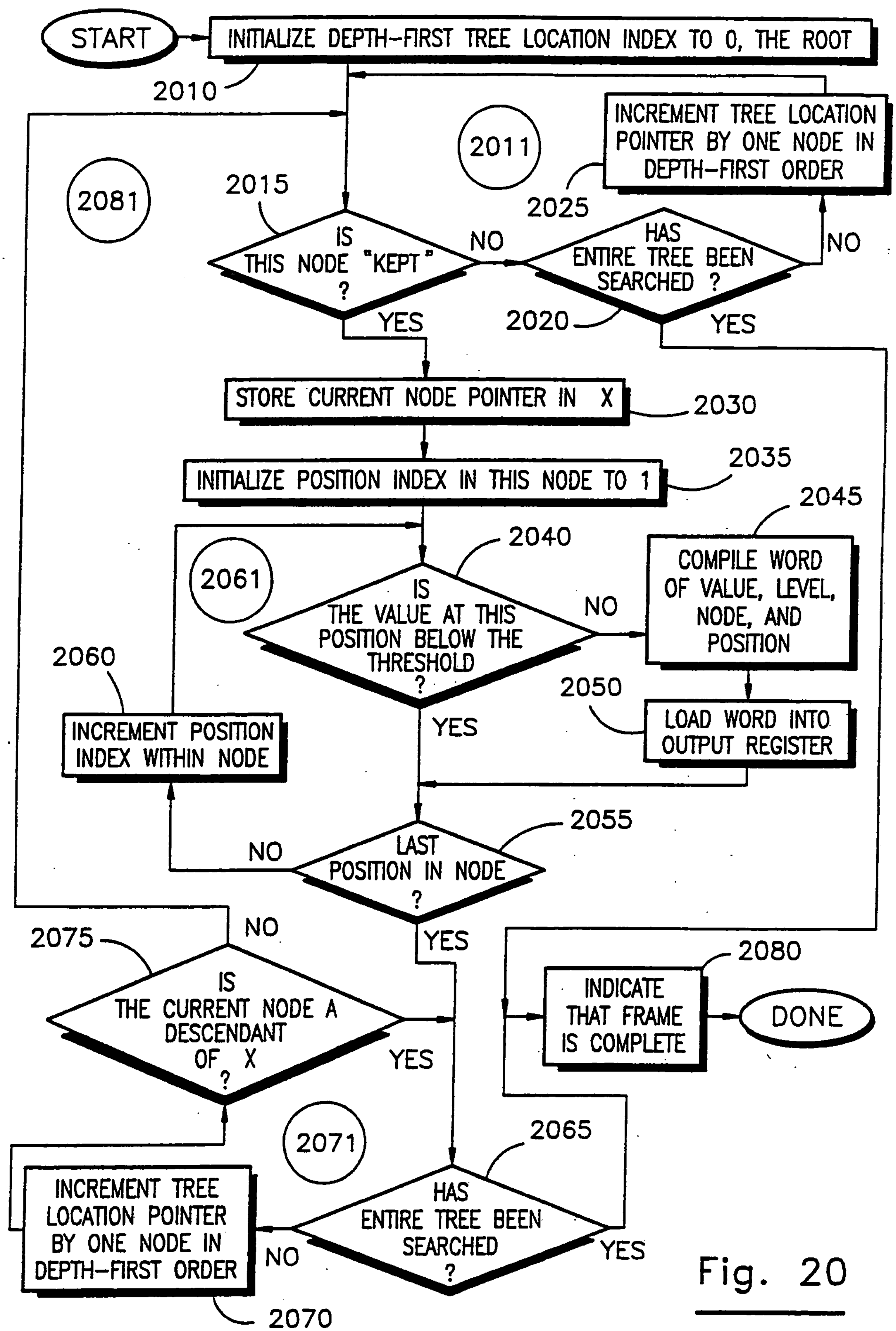

START
INITIALIZE DEPTH-FIRST TREE LOCATION INDEX TO 0, THE ROOT
2010
2011
INCREMENT TREE LOCATION POINTER BY ONE NODE IN DEPTH-FIRST ORDER
2081
2015
2025
IS THIS NODE "KEPT" ?
NO
HAS ENTIRE TREE BEEN SEARCHED ?
NO
YES
2020
YES
STORE CURRENT NODE POINTER IN X
2030
INITIALIZE POSITION INDEX IN THIS NODE TO 1
2035
2045
2040
2061
COMPILE WORD OF VALUE, LEVEL, NODE, AND POSITION
IS THE VALUE AT THIS POSITION BELOW THE THRESHOLD ?
NO
2060
2050
YES
LOAD WORD INTO OUTPUT REGISTER
INCREMENT POSITION INDEX WITHIN NODE
2055
NO
LAST POSITION IN NODE ?
NO
YES
2075
2080
IS THE CURRENT NODE A DESCENDANT OF X ?
INDICATE THAT FRAME IS COMPLETE
DONE
YES
2071
2065
INCREMENT TREE LOCATION POINTER BY ONE NODE IN DEPTH-FIRST ORDER
HAS ENTIRE TREE BEEN SEARCHED ?
NO
YES
2070

Fig. 20

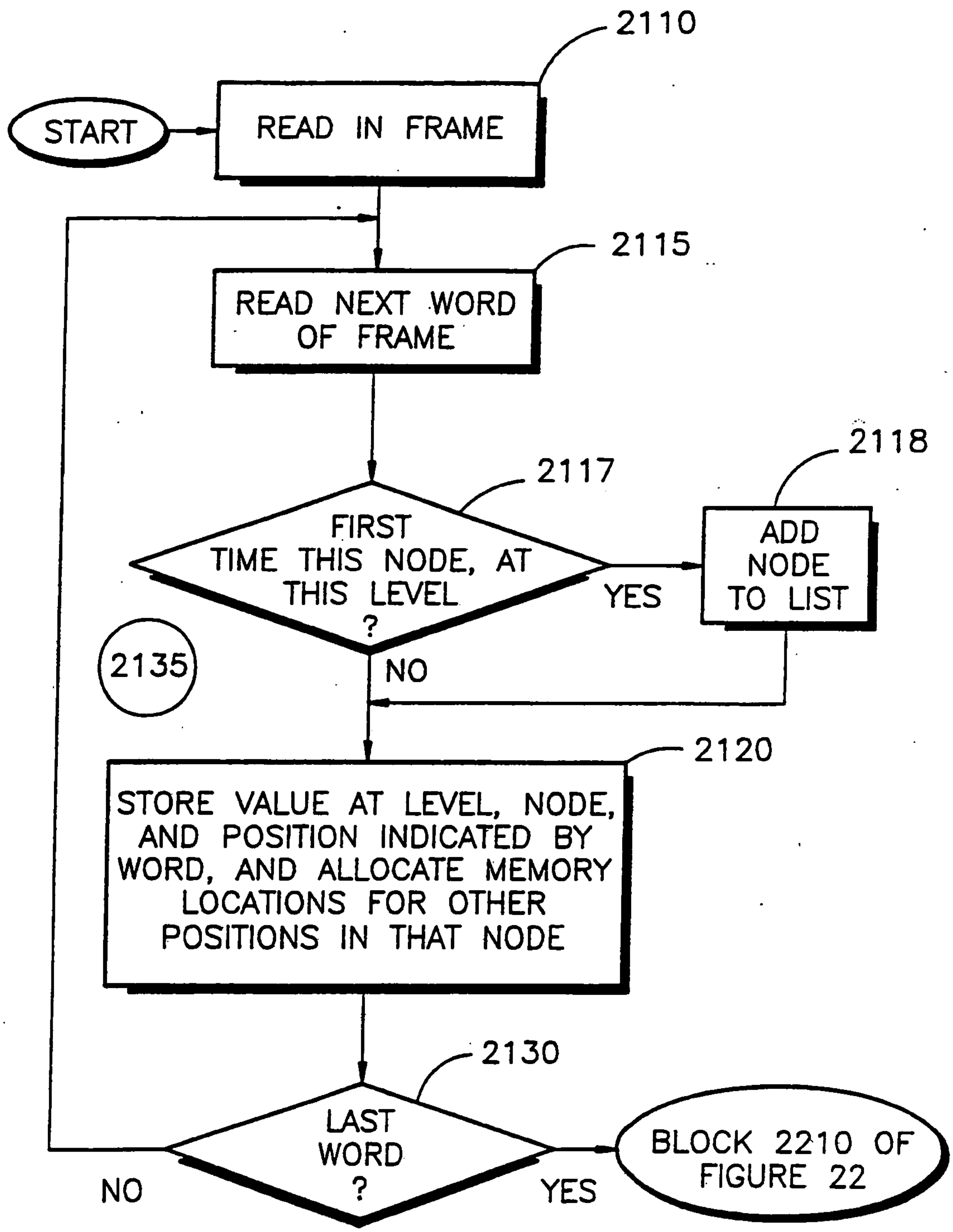

2110
START
READ IN FRAME
2115
READ NEXT WORD OF FRAME
2117
FIRST TIME THIS NODE, AT THIS LEVEL ?
YES
2118
ADD NODE TO LIST
2135
NO
2120
STORE VALUE AT LEVEL, NODE, AND POSITION INDICATED BY WORD, AND ALLOCATE MEMORY LOCATIONS FOR OTHER POSITIONS IN THAT NODE
2130
LAST WORD ?
NO
YES
BLOCK 2210 OF FIGURE 22

Fig. 21

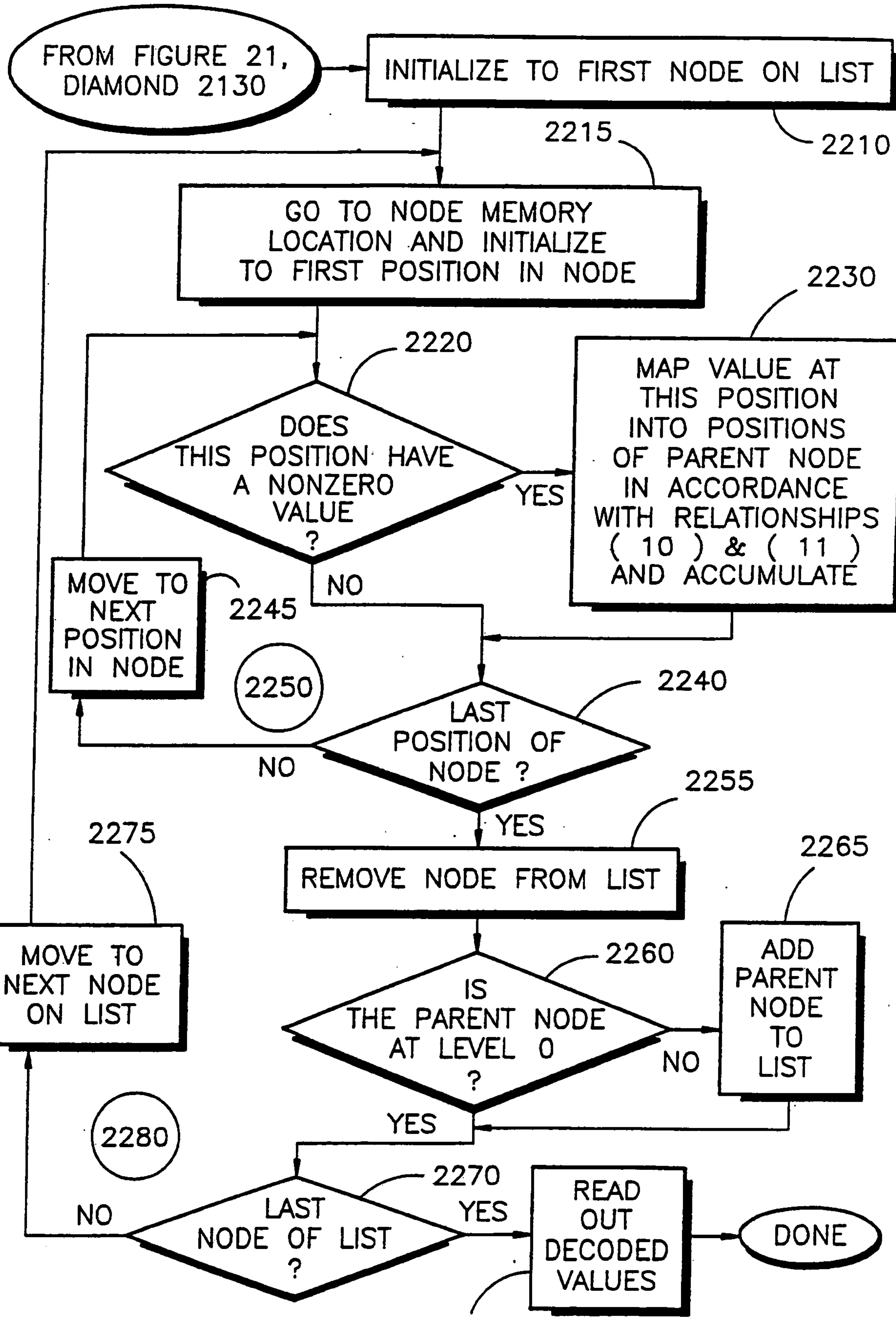

FROM FIGURE 21, DIAMOND 2130
INITIALIZE TO FIRST NODE ON LIST
2210
2215
GO TO NODE MEMORY LOCATION AND INITIALIZE TO FIRST POSITION IN NODE
2230
2220
DOES THIS POSITION HAVE A NONZERO VALUE ?
YES
MAP VALUE AT THIS POSITION INTO POSITIONS OF PARENT NODE IN ACCORDANCE WITH RELATIONSHIPS (10) & (11) AND ACCUMULATE
NO
MOVE TO NEXT POSITION IN NODE
2245
2250
2240
LAST POSITION OF NODE ?
NO
YES
2255
REMOVE NODE FROM LIST
2275
2265
MOVE TO NEXT NODE ON LIST
2260
IS THE PARENT NODE AT LEVEL 0 ?
NO
ADD PARENT NODE TO LIST
YES
2280
2270
LAST NODE OF LIST ?
NO
YES
READ OUT DECODED VALUES
DONE
2290

Fig. 22

INTERNATIONAL SEARCH REPORT

International Application No PCT/US91/03504

I. CLASSIFICATION OF SUBJECT MATTER (if several classification symbols apply, indicate all) [3]

According to International Patent Classification (IPC) or to both National Classification and IPC

IPC(5): G06G 7/00
US CL: 364/807

II. FIELDS SEARCHED

Minimum Documentation Searched [4]

Classification System	Classification Symbols
US	364/807,826,715.1,724.14,724.12,725,728.01,728.03,421 358/261.3,262.1,432,426 73/625,628 367/38,59 128/660.01

Documentation Searched other than Minimum Documentation to the Extent that such Documents are Included in the Fields Searched [5]

III. DOCUMENTS CONSIDERED TO BE RELEVANT [14]

Category *	Citation of Document, [16] with indication, where appropriate, of the relevant passages [17]	Relevant to Claim No. [18]
A	US, A, 4,706,499 (ANDERSON), 17 NOV. 1987)	1-30
A	US, A, 4,922,465 (PIEPRZAK ET AL) 1 MAY 1990	1-30
A,P	US, A, 5,000,183 (BONNEFOUS) 19 MAR 1991	1-30
Y,P	US, A, 5,014,134 (LAWTON ET AL) 7 MAY 1991 See Figure 1	1-30
Y,P	US, A, 4,974,187 (LAWTON) 27 NOV. 1990 See Abstract and Fig. 12	1-30
Y	I. Darbechies, "Oxthonormul Bases of Compactly Supported Wavelets", Comm. Pure, Applied Math, XL1 1988. See whole document.	1-30

* Special categories of cited documents: [15]

"A" document defining the general state of the art which is not considered to be of particular relevance

"E" earlier document but published on or after the international filing date

"L" document which may throw doubts on priority claim(s) or which is cited to establish the publication date of another citation or other special reason (as specified)

"O" document referring to an oral disclosure, use, exhibition or other means

"P" document published prior to the international filing date but later than the priority date claimed

"T" later document published after the international filing date or priority date and not in conflict with the application but cited to understand the principle or theory underlying the invention

"X" document of particular relevance; the claimed invention cannot be considered novel or cannot be considered to involve an inventive step

"Y" document of particular relevance; the claimed invention cannot be considered to involve an inventive step when the document is combined with one or more other such documents, such combination being obvious to a person skilled in the art.

"&" document member of the same patent family

IV. CERTIFICATION

Date of the Actual Completion of the International Search [2]	Date of Mailing of this International Search Report [2]
27 Aug. 1991	01 OCT 1991
International Searching Authority [1]	**Signature of Authorized Officer [20]**
ISA/US	NGUYEN NGOC-HO Ngoc Ho Nguyen INTERNATIONAL DIVISION for Jim Trammell

III. DOCUMENTS CONSIDERED TO BE RELEVANT (CONTINUED FROM THE SECOND SHEET)		
Category *	Citation of Document, [16] with indication, where appropriate, of the relevant passages [17]	Relevant to Claim No [18]
Y	S. Mallat, "Review of Multifrequency Channel Decomposition of Images and Wavelet Models" Technical Report 412, Robotics Report 1/8, NYU(1988) See whole document.	1-30
Y	T. Meyer, Wavelets and Operators, Analysis at Urbana, volil, edited by E. Berkson,N.T. Peck and J. Uhl, Lordon Math, Society, Lecture Notes Series 137, 1989 See whole document.	1-30
Y	S.G. Mallat, "A Theory For Multiresoluction Signal Decomposition": The Wavelet Representation", IEEE Transactions on Pattern Analysis and Machine Intelligence.Vor.II, No.7, July 1989 See whole document.	1-30
Y	G. Strans, "Wavelets and Dilation Equations: A Brief Introduction", SIAM Review, August, 1989 See whole document.	1-30
Y,P	R.R. Loifman, "Wavelet Analysis and Signal Processing", IMA Volumes In Mathematics and Its Applications, Vol. 22, Springer Verlag, 1990 See whole document	1-30

FURTHER INFORMATION CONTINUED FROM THE SECOND SHEET

V. [X] OBSERVATIONS WHERE CERTAIN CLAIMS WERE FOUND UNSEARCHABLE [1]

This international search report has not been established in respect of certain claims under Article 17(2) (a) for the following reasons:

1. [X] Claim numbers 1-30, because they relate to subject matter [1] not required to be searched by this Authority, namely:

   Scientific and mathematical theories

2. [ ] Claim numbers ...., because they relate to parts of the international application that do not comply with the prescribed requirements to such an extent that no meaningful international search can be carried out [1], specifically:

3. [ ] Claim numbers ___________, because they are dependent claims not drafted in accordance with the second and third sentences of PCT Rule 6.4(a).

VI. [ ] OBSERVATIONS WHERE UNITY OF INVENTION IS LACKING [2]

This International Searching Authority found multiple inventions in this international application as follows:

1. [ ] As all required additional search fees were timely paid by the applicant, this international search report covers all searchable claims of the international application.
2. [ ] As only some of the required additional search fees were timely paid by the applicant, this international search report covers only those claims of the international application for which fees were paid, specifically claims:
3. [ ] No required additional search fees were timely paid by the applicant. Consequently, this international search report is restricted to the invention first mentioned in the claims; it is covered by claim numbers:
4. [ ] As all searchable claims could be searched without effort justifying an additional fee, the International Searching Authority did not invite payment of any additional fee.

Remark on Protest

[ ] The additional search fees were accompanied by applicant's protest.

[ ] No protest accompanied the payment of additional search fees.